

TILTING THEORY FOR TREES VIA STABLE HOMOTOPY THEORY

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ABSTRACT. We show that variants of the classical reflection functors from quiver representation theory exist in any abstract stable homotopy theory, making them available for example over arbitrary ground rings, for quasi-coherent modules on schemes, in the differential-graded context, in stable homotopy theory as well as in the equivariant, motivic, and parametrized variant thereof. As an application of these equivalences we obtain abstract tilting results for trees valid in all these situations, hence generalizing a result of Happel.

The main tools introduced for the construction of these reflection functors are homotopical epimorphisms of small categories and one-point extensions of small categories, both of which are inspired by similar concepts in homological algebra.

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1. INTRODUCTION

In [Gab72] Gabriel classified (up to Morita equivalence) connected hereditary representation-finite algebras over an algebraically closed field k by means of their associated quivers. More precisely, an algebra is of the above kind if and only if it is the path algebra kQ , where the graph of the quiver Q belongs to a certain list of explicit graphs (which is completely exhausted by the Dynkin diagrams of type ADE).

Later Bernšteĭn, Gel'fand, and Ponomarev [BGP73] gave an elegant proof of this same result based on *reflection functors*. Given a quiver Q and a vertex $q \in Q$ which is a source (no arrow ends at q) or a sink (no arrow starts at q), the reflection $Q' = \sigma_q Q$ of Q is the quiver obtained by reversing the orientations of all edges

adjacent to q . Associated to this reflection at the level of quivers, there is a reflection functor $\text{Mod}(kQ) \rightarrow \text{Mod}(kQ')$ between the abelian categories of representations of the respective path algebras kQ and kQ' . These reflection functors are not equivalences, but the deviation from this is well-understood (see §5 for a precise definition of these functors).

Happel [Hap87] showed that if the quiver contains no *oriented* cycles, then the total derived functors of these reflection functors induce exact equivalences

$$D(kQ) \xrightarrow{\Delta} D(kQ')$$

between the derived categories of the path-algebras. *The main aim of this paper is to show that for oriented trees this result is a formal consequence of stability alone and it hence has variants in many other contexts arising in algebra, geometry, and topology.* Let us make this more precise and then illustrate the added generality by mentioning some examples.

Recall that there are many different approaches to axiomatic homotopy theory including model categories, ∞ -categories, and derivators. Here we choose to work with *derivators*, a notion introduced by Heller [Hel88], Grothendieck [Gro90], Franke [Fra96], and others. A derivator is some kind of a minimal extension of a classical derived category or homotopy category to a framework with a powerful calculus of homotopy (co)limits and also homotopy Kan extensions. In this extension, homotopy (co)limit constructions are characterized by ordinary universal properties, making them accessible to elementary methods from category theory. A derivator is *stable* if it admits a zero object and if homotopy pushouts and homotopy pullbacks coincide. Stable derivators provide an enhancement of triangulated categories (see [Gro13]). For example associated to a ring R there is the stable derivator \mathcal{D}_R of unbounded chain complexes over R , enhancing the classical derived category $D(R)$.

An important construction at the level of derivators is given by shifting: given a derivator \mathcal{D} and a small category B , there is the derivator \mathcal{D}^B of coherent B -shaped diagrams in \mathcal{D} , which is stable as soon as \mathcal{D} is. Since a quiver Q is simply a graph and hence has an associated free category, this shifting operation can be applied to quivers. For example, if we consider the stable derivator \mathcal{D}_k of a field k , then the shifted derivator \mathcal{D}_k^Q is equivalent to the stable derivator \mathcal{D}_{kQ} associated to the path-algebra kQ .

The main aim of this paper is then to show that if Q is an oriented tree and if Q' is obtained from Q by an arbitrary reorientation, then *for every stable derivator* \mathcal{D} there is a natural equivalence of derivators

$$\mathcal{D}^Q \simeq \mathcal{D}^{Q'}.$$

In the terminology of [GŠ14] we thus show that Q and Q' are *strongly stably equivalent* (Theorem 9.11 and its corollaries). Choosing specific stable derivators, this gives us refined variants of the above-mentioned result of Happel. If we take the stable derivator \mathcal{D}_k of a field, then we obtain equivalences of derivators

$$\mathcal{D}_{kQ} \simeq \mathcal{D}_k^Q \simeq \mathcal{D}_k^{Q'} \simeq \mathcal{D}_{kQ'},$$

and, in particular, exact equivalences $D(kQ) \xrightarrow{\Delta} D(kQ')$ of the underlying triangulated categories.

However, the same result is also true for the derivator \mathcal{D}_R of a ring R , for the derivator \mathcal{D}_X of a (quasi-compact and quasi-separated) scheme X , for the derivator \mathcal{D}_A of a differential-graded algebra A , for the derivator \mathcal{D}_E of a (symmetric) ring spectrum E , and for other stable derivators arising for example in stable homotopy theory as well as in its equivariant, motivic, or parametrized variants. Moreover, these equivalences are natural with respect to exact morphisms and hence commute, in particular, with various restriction of scalar functors, induction and coinduction functors, and localizations and colocalizations (see [GŠ14, §5] for more examples of stable derivators and many references).

By a combinatorial argument in order to obtain arbitrary reorientations of trees it is enough to construct reflection functors at sources and sinks inducing such natural equivalences. We mimic the classical construction from algebra, however we have to adapt certain steps significantly to make them work in this more general context. Let $q_0 \in Q$ be a source in an oriented tree. The classical reflection functors are roughly obtained by taking the sum of all outgoing morphisms at the source, passing to the cokernel of this map, and then using the structure maps of the biproduct in order to obtain a representation of the reflected quiver (see §5 for more details).

The main reason why we have to work harder in this more general context is that the final, innocent looking step cannot be performed that easily with abstract coherent diagrams: in such a diagram, we cannot simply replace the projections of a biproduct by the corresponding inclusions. Instead this is achieved by passing to larger diagrams which encode the biproduct objects and all the necessary projection and injection maps simultaneously.

A first step towards this is obtained by encoding finite biproducts by means of certain n -cubes of length two, the biproduct object sitting in the center (see §4). For two summands the picture to have in mind is

$$\begin{array}{ccccc}
 0 & \longrightarrow & X & \longrightarrow & 0 \\
 \downarrow & & \downarrow & & \downarrow \\
 Y & \longrightarrow & X \oplus Y & \longrightarrow & Y \\
 \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & X & \longrightarrow & 0.
 \end{array}$$

Using the basic theory of strongly bicartesian n -cubes established in [GŠ14, §8], this can be generalized to the case of finitely many arguments – the problem still being that we cannot simply pass from the projections to the injections.

The easy but key observation to solve this is that in such biproduct diagrams all ‘length two morphisms’ are invertible. This suggests that these diagrams should arise as restrictions of similar diagrams where the shape is given by some kind of ‘invertible n -cube of length two’ coming with all the necessary structure maps in both directions. In order to make this precise, we introduce the concept of *homotopical epimorphism* (see §6), and verify that this indeed works if we consider the standalone cube (see §7).

The next step is then to inductively take into account the parts of the representations which lie outside of this n -cube. This is achieved by means of *one-point*

extensions (see §8). Setting up things in the correct way, we end up with a category which contains both the original quiver and the reflected one as subcategories, such that we understand which representations of this larger category come from the respective quivers. This shows finally that the two quivers are strongly stably equivalent (see §9).

This paper is a sequel to [GŠ14] which in turn builds on [Gro13] and [GPS13], and as such is part of a *formal study of stability* (similarly to [GPS12] which is a *formal study of the interaction of stability and monoidal structure*). The aim of these papers and its sequels is to develop a formal, stable calculus which is available in arbitrary abstract stable homotopy theories, including typical situations arising in algebra, geometry, and topology (like the ones mentioned above). We expect this calculus to be rather rich, and we will develop further aspects of it somewhere else.

The content of the sections is as follows. In §§2–3 we recall some basics on derivators and, in particular, stable derivators. In §4 we describe how to model finite biproducts in stable derivators by means of n -cubes. In §5 we recall the classical construction of reflection functors in representation theory of quivers, and describe the strategy on how to generalize this to abstract stable derivators. In §6 we introduce and study homotopical epimorphisms. Specializing a combinatorial detection criterion for homotopy exact squares from [GPS13] we also obtain such a criterion for homotopical epimorphisms which is crucial in later sections. In §7 we establish a key example of a homotopical epimorphism, allowing us to describe finite biproducts in stable derivators by ‘invertible n -cubes’. In §8 we introduce one-point extensions, and show that homotopical epimorphisms are stable under one-point extensions, in a way that we can control the essential images of the associated restriction functors. In §9 we assemble the above results to construct the reflection functors, and deduce that arbitrary reorientations of oriented trees yield strongly stably equivalent quivers.

2. REVIEW OF DERIVATORS

In this section and the following one we include a short review of derivators and stable derivators, mainly to fix some notation and to quote a few results which are of constant use in later sections. More details can be found in [Gro13], its sequel [GPS13], and the many references therein.

Let $\mathcal{C}at$ denote the 2-category of small categories, $\mathcal{C}AT$ the 2-category of not necessarily small categories. The category with one object and its identity morphism only is denoted by $\mathbb{1}$. Note that objects $a \in A$ correspond to functors $a: \mathbb{1} \rightarrow A$ under the natural isomorphism $A \cong A^{\mathbb{1}}$.

A **prederivator** is simply a 2-functor $\mathcal{D}: \mathcal{C}at^{\text{op}} \rightarrow \mathcal{C}AT$.¹ Morphisms of prederivators are pseudo-natural transformations and transformations of prederivators are modifications so that we obtain a 2-category \mathcal{PDER} of prederivators

¹Recall that $\mathcal{C}at^{\text{op}}$ is obtained from $\mathcal{C}at$ by reversing the orientation of the functors but not of the natural transformations. This reflects the fact that, following Heller [Hel88] and Franke [Fra96], our convention for derivators is based on *diagrams*. There is an alternative, but isomorphic approach based on *presheaves*, i.e., contravariant functors, in which case also the orientations of the natural transformations should be changed; see for example [Gro90, Cis03].

(see [Bor94]). The category $\mathcal{D}(A)$ is the category of **coherent A -shaped diagrams** in \mathcal{D} . If $u: A \rightarrow B$ is a functor, then we denote the **restriction functor** by $u^*: \mathcal{D}(B) \rightarrow \mathcal{D}(A)$.

In the case of a functor $a: \mathbb{1} \rightarrow A$ classifying an object, $a^*: \mathcal{D}(A) \rightarrow \mathcal{D}(\mathbb{1})$ is an **evaluation functor**, taking values in the **underlying category** $\mathcal{D}(\mathbb{1})$. If $f: X \rightarrow Y$ is a morphism in $\mathcal{D}(A)$, then its image is denoted by $f_a: X_a \rightarrow Y_a$. These evaluation functors allow us to assign to any coherent diagram $X \in \mathcal{D}(A)$ an **underlying (incoherent) diagram** $A \rightarrow \mathcal{D}(\mathbb{1})$. The resulting functor $\mathcal{D}(A) \rightarrow \mathcal{D}(\mathbb{1})^A$ however is, in general, far from being an equivalence, and coherent diagrams are hence not determined by their underlying diagrams, even not up to isomorphism. Nevertheless, frequently we draw coherent diagrams as usual and say that such a diagram has the form of or looks like its underlying diagram.

Without additional axioms we cannot perform any constructions. A *derivator* is a prederivator which ‘allows for a well-behaved calculus of Kan extensions’, satisfying key properties of the calculus available in model categories, ∞ -categories, other approaches to higher category theory, as well as in ordinary categories. This seemingly abstract concept turns out to capture many typical constructions in homological algebra and homotopy theory, including

- (i) suspensions, loops, (co)fibers, Baratt–Puppe sequences [Gro13],
- (ii) homotopy orbits and homotopy fixed points of actions of discrete groups, leading to (co)homology of groups in the algebraic context,
- (iii) homotopy (co)ends and homotopy tensor products of functors [GPS12],
- (iv) spectrifications of prespectrum objects,

and many more including the reflection functors as we show in this paper. The axiomatization of this calculus is as follows.

If a restriction functor $u^*: \mathcal{D}(B) \rightarrow \mathcal{D}(A)$ admits a left adjoint $u_!: \mathcal{D}(A) \rightarrow \mathcal{D}(B)$, then we refer to it as a **left Kan extension functor**. A right adjoint $u_*: \mathcal{D}(A) \rightarrow \mathcal{D}(B)$ is a **right Kan extension functor**. In the examples of interest these are really *homotopy Kan extension functors*. We nevertheless follow the established terminology for ∞ -categories and refer to the functors as *Kan extensions*, and there is no risk of confusion as ‘categorical’ Kan extensions are meaningless in the context of an abstract prederivator. In the special case that $B = \mathbb{1}$ is the terminal category and we hence consider the unique functor $\pi = \pi_A: A \rightarrow \mathbb{1}$, the functor $\pi_! = \text{colim}_A$ is a **colimit functor** and $\pi_* = \lim_A$ a **limit functor**.

To actually work with these Kan extensions, one encodes the pointwise formulas which are known to be satisfied in the examples of interest (see [ML98, X.3.1] for the classical case). To make this axiom precise, we consider the special cases of comma squares

$$(2.1) \quad \begin{array}{ccc} (u/b) & \xrightarrow{p} & A \\ \pi \downarrow & \nearrow & \downarrow u \\ \mathbb{1} & \xrightarrow{b} & B \end{array} \quad \begin{array}{ccc} (b/u) & \xrightarrow{q} & A \\ \pi \downarrow & \searrow & \downarrow u \\ \mathbb{1} & \xrightarrow{b} & B \end{array}$$

which come with canonical transformations $u \circ p \rightarrow b \circ \pi$ and $b \circ \pi \rightarrow u \circ q$.

Definition 2.2. A **derivator** is a prederivator $\mathcal{D}: \mathcal{Cat}^{\text{op}} \rightarrow \mathcal{CAT}$ with the following properties.

- (Der1) $\mathcal{D}: \mathcal{Cat}^{\text{op}} \rightarrow \mathcal{CAT}$ takes coproducts to products. In particular, $\mathcal{D}(\emptyset)$ is the terminal category.
- (Der2) For any $A \in \mathcal{Cat}$, a morphism $f: X \rightarrow Y$ is an isomorphism in $\mathcal{D}(A)$ if and only if the morphisms $f_a: X_a \rightarrow Y_a, a \in A$, are isomorphisms in $\mathcal{D}(\mathbb{1})$.
- (Der3) Each functor $u^*: \mathcal{D}(B) \rightarrow \mathcal{D}(A)$ has both a left adjoint $u_!$ and a right adjoint u_* .
- (Der4) For any functor $u: A \rightarrow B$ and any $b \in B$ the canonical transformation

$$\pi_! p^* \rightarrow \pi_! p^* u^* u_! \rightarrow \pi_! \pi^* b^* u_! \rightarrow b^* u_!$$

is an isomorphism as is the canonical transformation

$$b^* u_* \rightarrow \pi_* \pi^* b^* u_* \rightarrow \pi_* q^* u^* u_* \rightarrow \pi_* q^*.$$

The axioms (Der1) and (Der3) together imply that $\mathcal{D}(A)$ has small categorical coproducts and products, hence, in particular, initial objects and final objects. These are the only actual 1-categorical (co)limits which must exist in each derivator. We will add a few explanatory comments on (Der4) after the following list of examples.

- Examples 2.3.* (i) Any category \mathbf{C} gives rise to a **represented** prederivator $y(\mathbf{C})$ defined by $y(\mathbf{C})(A) := \mathbf{C}^A$. Its underlying category is equivalent to \mathbf{C} itself. This prederivator is a derivator if and only if \mathbf{C} is complete and cocomplete, in which case the functors $u_!, u_*$ are ordinary Kan extension functors.
- (ii) A *Quillen model category* \mathbf{C} (see e.g. [Qui67, Hov99]) with *weak equivalences* \mathbf{W} has an underlying **homotopy derivator** $\mathcal{H}o(\mathbf{C})$ defined by formally inverting the pointwise weak equivalences $\mathcal{H}o(\mathbf{C})(A) := (\mathbf{C}^A)[(\mathbf{W}^A)^{-1}]$. The underlying category of $\mathcal{H}o(\mathbf{C})$ is the homotopy category $\mathbf{C}[\mathbf{W}^{-1}]$ of \mathbf{C} , and the functors $u_!, u_*$ are derived versions of the functors of $y(\mathbf{C})$ (see [Cis03] for the general case and [Gro13] for an easy proof in the case of combinatorial model categories).
- (iii) Similarly, let \mathcal{C} be an ∞ -category in the sense of Joyal [Joy] and Lurie [Lur09] (see [Gro10] for an introduction). Then we obtain a prederivator $\mathcal{H}o(\mathcal{C})$ by sending A to the homotopy category of $\mathcal{C}^{N(A)}$ where $N(A)$ denotes the nerve of A . For a sketch proof that for a complete and cocomplete ∞ -category this yields a derivator, the **homotopy derivator** of \mathcal{C} , we refer to [GPS13].
- (iv) Given a derivator \mathcal{D} and $B \in \mathcal{Cat}$ we can define the **shifted derivator** \mathcal{D}^B by $\mathcal{D}^B(A) := \mathcal{D}(B \times A)$ (see [Gro13, Theorem 1.25]). Moreover, the **opposite derivator** \mathcal{D}^{op} is defined by $\mathcal{D}^{\text{op}}(A) := \mathcal{D}(A^{\text{op}})^{\text{op}}$. These two constructions satisfy many compatibilities with the previous three examples. Also shifting and the passage to opposites are compatible in the sense that we have $(\mathcal{D}^B)^{\text{op}} \cong (\mathcal{D}^{\text{op}})^{B^{\text{op}}}$.

For more specific examples of derivators we refer the reader to [GŠ14, Examples 5.5]. The existence of opposites of derivators implies that the duality principle applies to the theory of derivators. The shifting operation is of central importance in this paper. Given a derivator \mathcal{D} and $B \in \mathcal{Cat}$, then \mathcal{D}^B is the homotopy theory of coherent B -shaped diagrams in \mathcal{D} . We will later use this in the special case where $B = Q$ is a quiver.

With these examples in mind, we see that (Der4) expresses the idea that Kan extensions in derivators are pointwise. In the notation of the axiom and of (2.1), let us consider the derivator $\mathcal{D} = y(\mathbf{C})$ represented by a complete and cocomplete

category \mathbf{C} . Given a diagram $X: A \rightarrow \mathbf{C}$, the canonical maps in (Der4) are the isomorphisms

$$\operatorname{colim}_{(u/b)} X \circ p \xrightarrow{\cong} \operatorname{LKan}_u(X)_b \quad \text{and} \quad \operatorname{RKan}_u(X)_b \xrightarrow{\cong} \lim_{(b/u)} X \circ q,$$

which allow us classically to compute Kan extensions in terms of colimits and limits.

A **morphism** of derivators is simply a morphism of underlying prederivators, i.e., a pseudo-natural transformation $F: \mathcal{D} \rightarrow \mathcal{E}$. Similarly, given two such morphisms $F, G: \mathcal{D} \rightarrow \mathcal{E}$, a **natural transformation** $F \rightarrow G$ is a modification. Thus, we define the 2-category \mathcal{DER} of derivators as a full sub-2-category of \mathcal{PDER} .

We now recall the notion of a *homotopy exact square* of small categories, which arguably is the main tool in the study of derivators as it allows us to extend many key facts about classical Kan extensions to the context of an abstract derivator and hence to [Examples 2.3](#) and in particular to [\[GŠ14, Examples 5.5\]](#). Given a derivator \mathcal{D} and a natural transformation living in a square

$$(2.4) \quad \begin{array}{ccc} D & \xrightarrow{p} & A \\ q \downarrow & \searrow_{\alpha} & \downarrow u \\ B & \xrightarrow{v} & C \end{array}$$

of small categories, we obtain **canonical mate-transformations**

$$(2.5) \quad q_! p^* \xrightarrow{\eta} q_! p^* u^* u_! \xrightarrow{\alpha^*} q_! q^* v^* v_! \xrightarrow{\epsilon} v^* u_! \quad \text{and}$$

$$(2.6) \quad u^* v_* \xrightarrow{\eta} p_* p^* u^* v_* \xrightarrow{\alpha^*} p_* q^* v^* v_* \xrightarrow{\epsilon} p_* q^*$$

Here, η denotes the adjunction units of the respective adjunctions and ϵ the respective adjunction counits. It can be shown that (2.5) is an isomorphism if and only if (2.6) is one.

A square (2.4) is by definition **homotopy exact** if the canonical mates (2.5) and (2.6) are isomorphisms. By (Der4) the comma squares (2.1) are homotopy exact, and many further examples can be established (see for example [\[Ayo, Mal11\]](#) and [\[Gro13, GPS13, GŠ14\]](#).) Here, we only collect the examples needed in this paper, and we refer to the above references for more details.

- Examples 2.7.* (i) *Kan extensions along fully faithful functors are again fully faithful.* If $u: A \rightarrow B$ is fully faithful, then the square $\operatorname{id} \circ u = \operatorname{id} \circ u$ is homotopy exact. Thus, the unit $\eta: \operatorname{id} \rightarrow u^* u_!$ and the counit $\epsilon: u^* u_* \rightarrow \operatorname{id}$ are isomorphisms ([\[Gro13, Proposition 1.20\]](#)).
- (ii) *Kan extensions and restrictions in unrelated variables commute.* For functors $u: A \rightarrow B$ and $v: C \rightarrow D$ the squares

$$\begin{array}{ccc} A \times C & \xrightarrow{u \times \operatorname{id}} & B \times C \\ \operatorname{id} \times v \downarrow & & \downarrow \operatorname{id} \times v \\ A \times D & \xrightarrow{u \times \operatorname{id}} & B \times D \end{array}$$

are homotopy exact ([\[Gro13, Proposition 2.5\]](#)).

- (iii) *Right adjoint functors are **homotopy final**.* If $u: A \rightarrow B$ is a right adjoint, then the square

$$\begin{array}{ccc} A & \xrightarrow{u} & B \\ \pi_A \downarrow & & \downarrow \pi_B \\ \mathbb{1} & \xrightarrow{\text{id}} & \mathbb{1} \end{array}$$

is homotopy exact, i.e., the canonical mate $\text{colim}_A u^* \rightarrow \text{colim}_B$ is an isomorphism ([Gro13, Proposition 1.18]).

- (iv) *Homotopy exact squares are compatible with pasting.* The passage to mate-transformation is functorial with respect to horizontal and vertical pasting. Consequently, horizontal and vertical pastings of homotopy exact squares are homotopy exact ([Gro13, Lemma 1.14]).

Since Kan extensions and restrictions in unrelated variables commute, there are *parametrized versions* of restrictions and Kan extensions. More precisely, associated to a derivator \mathcal{D} and a functor $u: A \rightarrow B$ there are adjunctions of derivators

$$(2.8) \quad (u_!, u^*): \mathcal{D}^A \rightleftarrows \mathcal{D}^B \quad \text{and} \quad (u^*, u_*): \mathcal{D}^B \rightleftarrows \mathcal{D}^A.$$

(See [Gro13, §2] for details on adjunctions of derivators.) In particular, if u is fully faithful, then the morphisms $u_!, u_*$ induce equivalences of derivators onto the respective essential images, a result we use without further reference in later sections.

Finally, we use the following notation. Given a (pre)derivator \mathcal{D} , then we write $X \in \mathcal{D}$ in order to indicate that $X \in \mathcal{D}(A)$ for some $A \in \mathcal{Cat}$.

3. STABLE DERIVATORS AND STRONGLY STABLY EQUIVALENT CATEGORIES

In this short section we recall some basics about pointed and stable derivators, mostly to fix some notation (again, for more details see [Gro13, GPS13]). Examples of such derivators arise from pointed or stable model categories and similarly for complete and cocomplete ∞ -categories. Thus stable derivators describe aspects of the calculus of homotopy Kan extensions available in such examples arising in algebra, geometry, and topology. We will also recall the notion of strongly stably equivalent categories introduced in [GŠ14].

As shown in [Gro13], the following definition is equivalent to the original one of Maltsiniotis [Mal07].

Definition 3.1. A derivator \mathcal{D} is **pointed** if $\mathcal{D}(\mathbb{1})$ has a zero object.

If \mathcal{D} is pointed then so are \mathcal{D}^B and \mathcal{D}^{op} . It follows that the categories $\mathcal{D}(A)$ have zero objects which are preserved by restriction and Kan extension functors.

In the pointed context, Kan extensions along inclusions of *cosieves* and *sieves* ‘extend diagrams by zero objects’. Recall that $u: A \rightarrow B$ is a **sieve** if it is fully faithful, and for any morphism $b \rightarrow u(a)$ in B , there exists an $a' \in A$ with $u(a') = b$. Dually, there is the notion of a **cosieve**, and we know already that Kan extensions along (co)sieves are fully faithful (see Examples 2.7).

Lemma 3.2 ([Gro13, Prop. 1.23]). *Let \mathcal{D} be a pointed derivator and let $u: A \rightarrow B$ be a sieve. Then $u_*: \mathcal{D}^A \rightarrow \mathcal{D}^B$ induces an equivalence onto the full subderivator of \mathcal{D}^B spanned by all diagrams $X \in \mathcal{D}^B$ such that X_b is zero for all $b \notin u(A)$.*

The functor u_* is **right extension by zero**. Dually, left Kan extensions along cosieves give **left extension by zero**.

In the framework of pointed derivators one can define **suspensions** and **loops**, **cofibers** and **fibers**, and similar constructions. In particular, we have adjunctions of derivators

$$(\Sigma, \Omega): \mathcal{D} \rightleftarrows \mathcal{D} \quad \text{and} \quad (\text{cof}, \text{fib}): \mathcal{D}^{[1]} \rightleftarrows \mathcal{D}^{[1]}$$

where $[1]$ is the category $(0 \rightarrow 1)$ with two objects and a unique non-identity morphism. Let us sketch the construction of the suspension Σ and the cofiber cof , the other two functors being dual. The commutative square $\square = [1]^2 = [1] \times [1]$,

$$(3.3) \quad \begin{array}{ccc} (0, 0) & \longrightarrow & (1, 0) \\ \downarrow & & \downarrow \\ (0, 1) & \longrightarrow & (1, 1), \end{array}$$

has full subcategories $i_\top: \top \rightarrow \square$ and $i_\bot: \bot \rightarrow \square$ obtained by removing $(1, 1)$ and $(0, 0)$, respectively. Since both inclusions are fully faithful, so are the associated Kan extension functors (see [Examples 2.7](#)). A square $Q \in \mathcal{D}(\square)$ is **cocartesian** if it lies in the essential image of $(i_\top)_!$. Dually, the **cartesian** squares are precisely the ones in the essential image of $(i_\bot)_*$. Given a coherent morphism $(f: X \rightarrow Y) \in \mathcal{D}([1])$, we obtain a cocartesian square in \mathcal{D} of the form

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow & & \downarrow \text{cof}(f) \\ 0 & \longrightarrow & Z \end{array}$$

first by right extending by zero and then by left Kan extension. A restriction of this cocartesian square along the obvious functor $[1] \rightarrow \square$ defines the cofiber of f . In the special case where $(f: X \rightarrow 0)$ itself is already obtained by right extension by zero, the cofiber object is called the suspension ΣX . Thus there is a defining cocartesian square looking like

$$\begin{array}{ccc} X & \longrightarrow & 0 \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & \Sigma X. \end{array}$$

A **(coherent) cofiber sequence** is a coherent diagram

$$\begin{array}{ccccc} X & \longrightarrow & Y & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & Z & \longrightarrow & W \end{array}$$

such that both squares are cocartesian and the two corners vanish as indicated. An arbitrary coherent morphism $(X \rightarrow Y) \in \mathcal{D}([1])$ can be extended to a cofiber sequence by a right extension by zero followed by a left Kan extension. (Note that this defines a cofiber sequence functor $\mathcal{D}^{[1]} \rightarrow \mathcal{D}^{[2] \times [1]}$ which is an equivalence onto its essential image.) Since the compound square is cocartesian ([\[GPS13, Corollary 4.10\]](#)), the object W is canonically isomorphic to ΣX . In particular, using this

isomorphism we can associate an underlying *incoherent cofiber sequence*

$$X \rightarrow Y \rightarrow Z \rightarrow \Sigma X \quad \text{in } \mathcal{D}(\mathbb{1})$$

to any coherent cofiber sequence.

Definition 3.4. A pointed derivator is **stable** if the classes of cocartesian squares and cartesian squares coincide.

The homotopy derivators of stable model categories or stable ∞ -categories are stable. If \mathcal{D} is stable then so is the shifted derivator \mathcal{D}^B ([Gro13, Proposition 4.3]) as is \mathcal{D}^{op} . Recall that a derivator is **strong** if it satisfies:

(Der5) For any A , the induced functor $\mathcal{D}(A \times [1]) \rightarrow \mathcal{D}(A)^{[1]}$ is full and essentially surjective.

This property does not play an essential role in the basic *theory* of derivators. It is mainly used if one wants to relate *properties* of stable derivators to the existence of *structure* on its values. The precise form of this axiom depends on the reference: in [Hel88] a stronger version is used. Represented derivators and homotopy derivators associated to model categories or ∞ -categories are strong as are shifts and opposites of strong derivators.

Theorem 3.5 ([Gro13, Theorem 4.16]). *Let \mathcal{D} be a strong, stable derivator. Then the categories $\mathcal{D}(A)$, $A \in \text{Cat}$, admit (canonical) triangulations.*

These triangulations are canonical in the following sense. Recall that a morphism between stable derivators is exact if it preserves zero object, cartesian and cocartesian squares. For more details on this notion see for example [Gro13].

Proposition 3.6 ([Gro13, Proposition 4.18]). *Let $F: \mathcal{D} \rightarrow \mathcal{E}$ be an exact morphism of strong, stable derivators. Then the components $F_A: \mathcal{D}(A) \rightarrow \mathcal{E}(A)$ can be endowed with the structure of an exact functor with respect to the triangulations of Theorem 3.5.*

Thus, like stable model categories and stable ∞ -categories, stable derivators provide an enhancement of triangulated categories. With this in mind, the following notion was introduced in [GS14].

Definition 3.7. Two small categories A and A' are **strongly stably equivalent**, in notation $A \stackrel{s}{\sim} A'$, if for every stable derivator \mathcal{D} there is an equivalence of derivators

$$\mathcal{D}^A \simeq \mathcal{D}^{A'}$$

which is natural with respect to exact morphisms of stable derivators. Such a natural equivalence is a **strong stable equivalence**.

For every field k there is the stable derivator \mathcal{D}_k associated to the projective model structure on the category of unbounded chain complexes over k . Recall that a module over the path-algebra kQ of a quiver Q is equivalently specified by a functor $Q \rightarrow \text{Mod}(k)$, where, strictly speaking, Q is the free category generated by the graph Q . This identification extends to chain complexes and implies that \mathcal{D}_k^Q is equivalent to the derivator \mathcal{D}_{kQ} of the path-algebra. In particular, for two strongly stably equivalent quivers Q and Q' there are natural, exact equivalences $D(kQ) \simeq D(kQ')$ of the underlying derived categories of the path-algebras, showing that the quivers are derived equivalent.

However, the statement that two categories or quivers are *strongly stably equivalent* is a much stronger statement as such equivalences then also have to exist for the derivator \mathcal{D}_R of a ring R , the derivator \mathcal{D}_X of a (quasi-compact and quasi-separated) scheme X , the derivator \mathcal{D}_A of a differential-graded algebra A , for the derivator \mathcal{D}_E of a (symmetric) ring spectrum E , and other examples arising for example in equivariant stable, motivic stable, or parametrized stable homotopy theory. Moreover, since strong stable equivalences are natural with respect to exact morphisms all the above equivalences commute with various restriction and (co)induction of scalar functors as well as Bousfield (co)localizations (see [GŠ14, §4] for many examples of stable derivators and references as well as further comments on Definition 3.7).

In order to show that certain categories or quivers are strongly stably equivalent we use suitable combinations of Kan extension morphisms as in (2.8). For that purpose it is convenient to recall the following lemma which allows us to ‘detect’ (co)cartesian squares in larger coherent diagrams.

Lemma 3.8 ([Gro13, Prop. 3.10]). *Suppose $u: C \rightarrow B$ and $v: \square \rightarrow B$ are functors, with v injective on objects, and let $b = v(1, 1) \in B$. Suppose furthermore that $b \notin u(C)$, and that the functor $\ulcorner \rightarrow (B \setminus b)/b$ induced by v has a left adjoint. Then for any derivator \mathcal{D} and any $X \in \mathcal{D}(C)$, the square $v^*u_!X$ is cocartesian.*

We will also need its generalization to arbitrary *cocones*. Given a small category A let A^\triangleright be the **cocone** on A , i.e., the category which is obtained from A by adjoining a new terminal object ∞ . The cocone construction $(-)^{\triangleright}$ is obviously functorial in A , and there is a natural fully faithful inclusion $i = i_A: A \rightarrow A^\triangleright$. A cocone $X \in \mathcal{D}^{A^\triangleright}$ is **colimiting** if it lies in the essential image of the left Kan extension functor $(i_A)_!: \mathcal{D}^A \rightarrow \mathcal{D}^{A^\triangleright}$.

Lemma 3.9 ([GPS13, Lemma 4.5]). *Let $A \in \mathcal{Cat}$, and let $u: C \rightarrow B$, $v: A^\triangleright \rightarrow B$ be functors. Suppose that there is a full subcategory $B' \subseteq B$ such that*

- (i) $u(C) \subseteq B'$ and $v(\infty) \notin B'$;
- (ii) $vi: A \rightarrow B$ factors through the inclusion $B' \subseteq B$; and
- (iii) the functor $A \rightarrow B'/v(\infty)$ induced by v has a left adjoint.

*Then for any derivator \mathcal{D} and any $X \in \mathcal{D}(C)$, the diagram $v^*u_!X$ is in the essential image of $i_!$. In particular, $(v^*u_!X)_\infty$ is the colimit of $i^*v^*u_!X$.*

4. FINITE BIPRODUCTS VIA n -CUBES

For the construction of abstract reflection functors in §9 we will need a description of finite biproducts in stable derivators by means of certain coherent diagrams. For this purpose, in this section we briefly recall from [Gro13, §4.1] the construction of the biproduct of pairs of objects in stable derivators. Then, using basic results on strongly bicartesian n -cubes from [GŠ14, Section 8], we extend this description to the case of finitely many summands.

Let \mathcal{D} be a stable derivator and let $[2]$ be the category $(0 < 1 < 2)$. For the case of two summands, we consider the category $[2] \times [2]$ and the following *full* subcategories of $[2] \times [2]$ which we define by listing their objects,

- (i) $A_1: (1, 2)-(2, 1)$,
- (ii) $A_2: (1, 2)-(2, 2)-(2, 1)$, and
- (iii) $A_3: (0, 2)-(1, 2)-(2, 2)-(2, 1)-(2, 0)$.

There are obvious inclusions $j_1: A_1 \rightarrow A_2$, $j_2: A_2 \rightarrow A_3$, and $j_3: A_3 \rightarrow [2] \times [2]$, and the following sequence of Kan extensions

$$(4.1) \quad \mathcal{D}^{A_1} \xrightarrow{(j_1)_*} \mathcal{D}^{A_2} \xrightarrow{(j_2)!} \mathcal{D}^{A_3} \xrightarrow{(j_3)_*} \mathcal{D}^{[2] \times [2]}$$

consists of fully faithful morphisms of derivators (Examples 2.7). Moreover, since j_1 is the inclusion of a sieve it follows from Lemma 3.2 that $(j_1)_*$ is right extension by zero. Similarly, j_2 is the inclusion of a cosieve and hence $(j_2)!$ is left extension by zero. Finally, a repeated application of Lemma 3.8 implies that $(j_3)_*$ amounts to forming four bicartesian squares. We summarize this construction in the following lemma (see [Gro13, §4.1] for more details).

Lemma 4.2. *Let \mathcal{D} be a stable derivator and let $\mathcal{D}^{[2] \times [2], \text{ex}} \subseteq \mathcal{D}^{[2] \times [2]}$ be the full subderivator spanned by the coherent diagrams vanishing on the corners and making all squares bicartesian. Then (4.1) induces a natural equivalence $\mathcal{D}^{\mathbb{1} \sqcup \mathbb{1}} \simeq \mathcal{D}^{[2] \times [2], \text{ex}}$.*

We only have to verify that a diagram in the image of (4.1) also vanishes at the corner $(0,0)$. Note that (Der1) implies that there is a canonical equivalence $\mathcal{D} \times \mathcal{D} \simeq \mathcal{D}^{\mathbb{1} \sqcup \mathbb{1}}$. By the above, (4.1) sends $(X, Y) \in \mathcal{D} \times \mathcal{D} \simeq \mathcal{D}^{\mathbb{1} \sqcup \mathbb{1}} = \mathcal{D}^{A_1}$ to a coherent diagram in $\mathcal{D}^{[2] \times [2]}$ looking like

$$(4.3) \quad \begin{array}{ccccc} Z & \longrightarrow & \tilde{X} & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow \\ \tilde{Y} & \longrightarrow & B & \longrightarrow & Y \\ \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & X & \longrightarrow & 0. \end{array}$$

Since all squares in this diagram are bicartesian, since composites of bicartesian squares are again bicartesian ([GPS13, Corollary 4.10]) and since pullbacks of isomorphisms are again isomorphisms ([Gro13, Proposition 3.12]), we obtain isomorphisms

$$X \cong \tilde{X}, \quad Y \cong \tilde{Y}, \quad \text{and} \quad Z \cong 0.$$

Moreover, as a pullback over a zero object is a product and dually ([GPS13, Corollary 4.11]), it follows that B is a biproduct of X and Y .

The diagram (4.3) actually is a coherent version of the biproduct object together with the corresponding inclusions and projections. Thus, Lemma 4.2 can be read as saying that for a stable derivator \mathcal{D} , the derivator $\mathcal{D}^{\mathbb{1} \sqcup \mathbb{1}} \simeq \mathcal{D} \times \mathcal{D}$ is naturally equivalent to the *derivator of biproduct diagrams*. We want to emphasize that in (4.3) all ‘length two morphisms’

$$(i, 0) \rightarrow (i, 2) \quad \text{and} \quad (0, j) \rightarrow (2, j)$$

are sent to isomorphisms.

We now discuss a variant of this construction for biproducts of finitely many summands. For this purpose we recall from [GS14, §8] some basic notation and terminology related to n -cubes in derivators. We denote the n -cube by $[1]^n = [1] \times \dots \times [1]$. Note that there is an order-preserving isomorphism between $[1]^n$ and the power set of $\{1, \dots, n\}$, and we allow ourselves to pass back and forth between these two descriptions without explicit mention. For $0 \leq k \leq n$ we denote by $i_{\geq k}: [1]_{\geq k}^n \rightarrow [1]^n$ the inclusion of the full subcategory spanned by all subsets of

cardinality at least k . Of course, there are obvious variants of these subcategories which will be denoted similarly. In particular, the full subcategory $[1]_{=n-1}^n \subseteq [1]^n$ is the *discrete* category $n \cdot \mathbb{1} = \mathbb{1} \sqcup \dots \sqcup \mathbb{1}$ with n objects.

Definition 4.4. Let \mathcal{D} be a derivator. An n -cube $X \in \mathcal{D}^{[1]^n}$ is **strongly cartesian** if it lies in the essential image of $(i_{\geq n-1})_*: \mathcal{D}^{[1]_{\geq n-1}^n} \rightarrow \mathcal{D}^{[1]^n}$. An n -cube X is **cartesian** if it lies in the essential image of $(i_{\geq 1})_*$.

Since $i_{\geq k}$ is fully faithful, the same is true for $(i_{\geq k})_*$, and an n -cube X lies in the essential image of $(i_{\geq k})_*$ if and only if the unit $\eta: X \rightarrow (i_{\geq k})_*(i_{\geq k})^*(X)$ is an isomorphism. In particular, X is strongly cartesian if and only if the unit $X \rightarrow (i_{\geq n-1})_*(i_{\geq n-1})^*(X)$ is an isomorphism.

Inspired by the work of Goodwillie [Goo92], in [GŠ14, §8] we have shown that an n -cube in a derivator is strongly cartesian if and only if all *subcubes* are cartesian (see [GŠ14, Theorem 8.3]) if and only if all *subsquares* are cartesian (see [GŠ14, Corollary 8.12]). This implies the equivalence of the third and the fourth characterization in the following theorem.

Theorem 4.5 ([GPS13, Theorem 7.1], [GŠ14, Corollary 8.11]). *The following are equivalent for a pointed derivator \mathcal{D} .*

- (i) *The adjunction $(\Sigma, \Omega): \mathcal{D}(\mathbb{1}) \rightarrow \mathcal{D}(\mathbb{1})$ is an equivalence.*
- (ii) *The adjunction $(\text{cof}, \text{fib}): \mathcal{D}([1]) \rightarrow \mathcal{D}([1])$ is an equivalence.*
- (iii) *The derivator \mathcal{D} is stable.*
- (iv) *An n -cube in \mathcal{D} , $n \geq 2$, is strongly cartesian if and only if it is strongly cocartesian.*

An n -cube which is simultaneously strongly cartesian and strongly cocartesian is **strongly bicartesian**. In the case of $n = 2$ this reduces to the classical notion of a **bicartesian square**. Similar to bicartesian squares also strongly bicartesian n -cubes enjoy a 2-out-of-3 property with respect to composition and cancellation (see [GŠ14, §8] for the case of n -cubes).

With the construction of coherent diagrams for finite biproducts in mind, let us consider the functors

$$(4.6) \quad n \cdot \mathbb{1} = [1]_{=n-1}^n \xrightarrow{i_1} [1]_{\geq n-1}^n \xrightarrow{i_2} [1]^n \xrightarrow{i_3} I \xrightarrow{i_4} [2]^n.$$

Here i_1 and $i_2 = i_{\geq n-1}$ are the obvious fully faithful inclusions and $i_4 i_3: [1]^n \rightarrow [2]^n$ is the inclusion as the n -cube

$$[(1, \dots, 1), (2, \dots, 2)] = [1, 2] \times \dots \times [1, 2].$$

Finally, $I \subseteq [2]^n$ is the full subcategory spanned by $[(1, \dots, 1), (2, \dots, 2)]$ and the n additional corners

$$(4.7) \quad (0, 2, 2, \dots, 2, 2), \quad (2, 0, 2, 2, \dots, 2, 2), \quad \dots, \quad (2, 2, \dots, 2, 2, 0),$$

while i_3 and i_4 are the fully faithful inclusions arising from the obvious factorization. Since all four functors are fully faithful the same is true for the Kan extension functors

$$(4.8) \quad \mathcal{D}^{n \cdot \mathbb{1}} = \mathcal{D}^{[1]_{=n-1}^n} \xrightarrow{(i_1)^*} \mathcal{D}^{[1]_{\geq n-1}^n} \xrightarrow{(i_2)^*} \mathcal{D}^{[1]^n} \xrightarrow{(i_3)!} \mathcal{D}^I \xrightarrow{(i_4)^*} \mathcal{D}^{[2]^n}.$$

We show next that in the case of a stable derivator \mathcal{D} the essential image consists of the full subderivator $\mathcal{D}^{[2]^n, \text{ex}} \subseteq \mathcal{D}^{[2]^n}$ spanned by the coherent diagrams such that

- (i) all subcubes are strongly bicartesian,
- (ii) the values at all corners are trivial, and
- (iii) the maps $(i_1, \dots, i_{k-1}, 0, i_{k+1}, \dots, i_n) \rightarrow (i_1, \dots, i_{k-1}, 2, i_{k+1}, \dots, i_n)$ are sent to isomorphisms for all $i_1, \dots, i_{k-1}, i_{k+1}, \dots, i_n$ and k .

The proposition will justify the claim that $\mathcal{D}^{[2]^n, \text{ex}}$ is a derivator. Note that property (iii) is a consequence of (i) and (ii) but we emphasize it here as this property will prove important later on.

Proposition 4.9. *Let \mathcal{D} be a stable derivator. Then (4.8) consists of fully faithful functors and induces an equivalence $\mathcal{D}^{n, \mathbb{1}} \rightarrow \mathcal{D}^{[2]^n, \text{ex}}$ of derivators which is natural with respect to exact morphisms.*

Proof. Since $i_1: [1]_{=n-1}^n \rightarrow [1]_{\geq n-1}^n$ is the inclusion of a sieve, $(i_1)_*$ is right extension by zero (Lemma 3.2). It follows that $(i_1)_*: \mathcal{D}^{[1]_{=n-1}^n} \rightarrow \mathcal{D}^{[1]_{\geq n-1}^n}$ induces an equivalence on the full subderivator spanned by diagrams satisfying this vanishing condition. By definition of strongly cartesian n -cubes, the composition $(i_2)_*(i_1)_*: \mathcal{D}^{[1]_{=n-1}^n} \rightarrow \mathcal{D}^{[1]^n}$ induces an equivalence onto the full subderivator of strongly cartesian n -cubes which vanish on the final object. Note next that $i_3: [1]^n \rightarrow I$ is the inclusion of a cosieve. Hence $(i_3)_!: \mathcal{D}^{[1]^n} \rightarrow \mathcal{D}^I$ is left extension by zero and induces an equivalence onto the full subderivator of \mathcal{D}^I spanned by the diagrams which vanish on the corners (4.7) (again by Lemma 3.2). This equivalence restricts further to an equivalence of the respective subderivators spanned by diagrams vanishing on the final object and making the n -cube strongly cartesian. It remains to understand the effect of $(i_4)_*$. It follows from a repeated application of Lemma 3.9 that $(i_4)_*: \mathcal{D}^I \rightarrow \mathcal{D}^{[2]^n}$ amounts to adding $2^n - 1$ strongly bicartesian n -cubes. The verification of this is a straightforward higher-dimensional variant of the corresponding verifications in dimension two as carried out in [Gro13, §4.1], and we leave the details to the reader.

As an upshot, the composition (4.8) induces an equivalence onto the full subderivator of $\mathcal{D}^{[2]^n}$ spanned by the diagrams making all n -cubes strongly bicartesian (use [GŠ14, Corollary 8.11]) and vanishing on the corners (4.7) as well as on the final vertex. Note that this implies that all subsquares are then also bicartesian ([GŠ14, Corollary 8.10]). As isomorphisms in coherent diagrams are stable under pullbacks, it follows from the vanishing conditions of the diagrams that also condition (iii) in the definition of $\mathcal{D}^{[2]^n, \text{ex}}$ is satisfied, and hence that these diagrams actually vanish on all corners. Finally, the fact that this equivalence is natural with respect to exact morphisms is a consequence of [GŠ14, Theorem 4.6]. \square

Note that the objects of $\mathcal{D}^{[2]^n, \text{ex}}$ are coherent diagrams for finite biproducts $X_1 \oplus \dots \oplus X_n$ which also encode all partial biproducts together with all the inclusion and projection maps. Thus, the proposition makes precise the expected result that in the stable case the derivator $\mathcal{D}^{n, \mathbb{1}} \simeq \mathcal{D} \times \dots \times \mathcal{D}$ is naturally equivalent to the derivator $\mathcal{D}^{[2]^n, \text{ex}}$ of n -fold biproduct diagrams. We will use a variant of this observation in our construction of abstract reflection functors in §9.

5. REFLECTION FUNCTORS IN REPRESENTATION THEORY

The main aim of this paper is to construct reflection functors for arbitrary stable derivators, and to show that they give rise to strongly stably equivalent quivers. We begin by recalling some details about the classical construction at the level of

abelian categories of representations. By a quiver Q we formally mean a quadruple (Q_0, Q_1, s, t) consisting of a set of vertices Q_0 , a set of arrows Q_1 , and two maps $s, t: Q_1 \rightarrow Q_0$ assigning to each arrow its source and target vertex, respectively. An **oriented tree** is a connected quiver without unoriented cycles. In other words, an oriented tree is obtained from an ordinary tree by equipping each edge with an orientation. All our quivers in the sequel will be finite; that is Q_0 and Q_1 will be finite sets.

Given a quiver Q and a vertex $q \in Q$, there is the **reflected quiver** $\sigma_q Q$ obtained from Q by changing the orientations of all arrows adjacent to q . A vertex q_0 of a quiver is a **sink** or a **source** if all edges adjacent to it have q_0 as their target or source, respectively.

These reflections at sources and sinks and the associated reflection functors between the respective module categories have played an important role in representation theory, e.g., in the classification of quivers of finite representation type (see for example [Gab72, BGP73, Hap86]). For more details about tilting theory itself we refer to [AHHK07].

The classical construction of the reflection functors in the case of a source q_0 is as follows. Let $Q' = \sigma_{q_0} Q$ be the reflected quiver and let R be an arbitrary ground ring. If $M: Q \rightarrow \text{Mod}(R)$ is a representation, then the **reflection functor**

$$s_{q_0}^-: \text{Mod}(RQ) \rightarrow \text{Mod}(RQ')$$

sends M to the following representation $M' = s_{q_0}^-(M): Q' \rightarrow \text{Mod}(R)$. On vertices we have $M'_{q'} = M_{q'}$ for all $q' \neq q_0$, and similarly $M'_f = M_f$ on all edges which are not adjacent to q_0 . If $q_0 \rightarrow q_i, 1 \leq i \leq n$, is the set of morphisms adjacent to q_0 in Q , then M'_{q_0} is defined as

$$(5.1) \quad M'_{q_0} = \text{coker}(M_{q_0} \rightarrow \bigoplus_{i=1, \dots, n} M_{q_i}).$$

The structure maps $M'_{q_j} \rightarrow M'_{q_0}$ in the reflected direction are given by

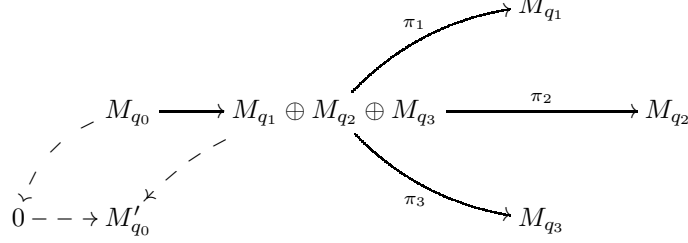
$$(5.2) \quad M_{q_j} \xrightarrow{\iota_j} \bigoplus_{i=1, \dots, n} M_{q_i} \rightarrow M'_{q_0},$$

where the undecorated morphism is the canonical map to the cokernel. It is easy to use the universal property of the cokernel to define this reflection functor $s_{q_0}^-$ on morphisms of representations. The case of a sink is dual. In particular, since q_0 is a sink in Q' , we obtain a functor

$$s_{q_0}^+: \text{Mod}(RQ') \rightarrow \text{Mod}(RQ)$$

which is easily checked to be a right adjoint to $s_{q_0}^-$.

Thus the situation is roughly summarized by the following diagram in which for the sake of simplicity we suppressed any additional branches which could potentially be attached to the vertices $M_{q_i}, i \geq 1$:



The adjunction $(s_{q_0}^-, s_{q_0}^+): \text{Mod}(RQ) \rightleftarrows \text{Mod}(RQ')$ is *not* an equivalence of abelian categories of representations (for more detail see e.g. [ASS06]). However, the following result was established by Happel.

Theorem 5.3 ([Hap87]). *Let Q be a quiver without oriented cycles and let $q_0 \in Q$ be a source. Then over a field k , the reflection functors $s_{q_0}^-$ and $s_{q_0}^+$ induce a pair of inverse exact equivalences*

$$\mathbf{L}s_{q_0}^- : D(kQ) \rightleftarrows D(kQ') : \mathbf{R}s_{q_0}^+$$

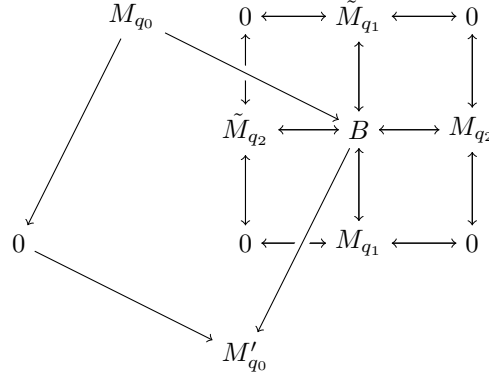
of derived categories.

Remark 5.4. Let us briefly recall how the theorem has been historically proved. While attempting to generalize the reflection functors, Auslander, Platzeck and Reiten noted in [APR79] that there exists a certain bimodule ${}_{kQ'}T_{kQ}$ such that $s_{q_0}^+$ is naturally equivalent to $\text{hom}_{kQ'}(T, -)$. Consequently, *tilting functors* were defined by Brenner and Butler in [BB80] and *tilting modules* by Happel and Ringel in [HR82]. In their terminology, $(T \otimes_{kQ} -, \text{hom}_{kQ'}(T, -))$ is a pair of tilting functors and ${}_{kQ'}T$ and T_{kQ} are tilting modules.

Now the crucial observation due to Happel [Hap87] is that the derived functor $\mathbf{R}\text{hom}_{kQ'}(T, -)$ induces an (exact) equivalence between the bounded derived categories $D^b(kQ)$ and $D^b(kQ')$. Since the latter categories are precisely the categories of compact objects of $D(kQ')$ and $D(kQ)$, respectively, an easy abstract argument shows that $\mathbf{R}\text{hom}_{kQ'}(T, -)$ must also be an equivalence between $D(kQ')$ and $D(kQ)$. Finally, $(s_{q_0}^-, s_{q_0}^+)$ is checked to be a Quillen adjunction with respect to the projective model structure on $\text{Ch}(kQ)$ (see [Hov99, Definition 2.3.3 and Theorem 2.3.11]) and the injective model structure on $\text{Ch}(kQ')$ (see [Hov99, Definition 2.3.12 and Theorem 2.3.13]). In combination with the result of Happel this implies that $(s_{q_0}^-, s_{q_0}^+)$ even is a Quillen equivalence.

In this paper we obtain for oriented trees a generalization of this result which is valid in the context of an arbitrary stable derivator. More precisely, we establish as [Theorem 9.11](#) that the quivers Q and Q' are not only derived equivalent but actually *strongly stably equivalent*. In particular, this implies variants of Happel's result for representations over an arbitrary ground ring, for quasi-coherent modules on schemes, in the differential-graded context, and in the spectral context. Moreover, these equivalences are natural with respect to exact morphisms (for more comments on the added generality see [GŠ14, Section 3]).

The tricky point in the construction of similar reflection functors for abstract stable derivators (and hence in the context of homotopy coherent diagrams) is that one cannot simply take a morphism in such a diagram and replace it by a map in


 FIGURE 1. Reflection functors using ‘invertible n -cubes’

the opposite direction. Note that this is what we do in the classical case. First, in the definition of M'_{q_0} we use that $M_{q_1} \oplus \dots \oplus M_{q_n}$ enjoys the universal property of a *product* (see (5.1)). Then we use that it also is a *coproduct* coming with canonical inclusion maps which allows us to define M' on edges (see (5.2)). In order to mimic this step of ‘replacing’ the projections of the biproduct by the inclusions in the context of abstract stable derivators we have to work a bit harder. The strategy consists of the following steps and is illustrated in Figure 1 (where again, for simplicity, we suppress additional branches of the tree).

- (i) Extend the quiver Q to a category by adding $[2]^n$, an n -cube of length two. At the level of representations this amounts to gluing in a representation of $[2]^n$ which is a finite biproduct diagram for $M_{q_1} \oplus \dots \oplus M_{q_n}$. This will be achieved by a variant of Proposition 4.9.
- (ii) As in Proposition 4.9, it will be true that the extended representations send all ‘length two morphisms’ in the n -cube $[2]^n$ to isomorphisms. This suggests that the representations should be restrictions of representations of the category obtained by formally inverting all such ‘length two morphisms’. Taking the cube $[2]^n$ alone, we show in §7 that this localization $q: [2]^n \rightarrow R^n$ is a *homotopical epimorphism* (a notion we introduce in §6), and hence induces a fully faithful restriction functor $q^*: \mathcal{D}^{R^n} \rightarrow \mathcal{D}^{[2]^n}$. The essential image of q^* consists precisely of the diagrams which invert all ‘length two morphisms’ (see Corollary 7.5 and Corollary 7.6).
- (iii) In order to check that this localization step interacts nicely with additional pieces of the representations attached to the cube, we introduce the concept of a *one-point extension* (see §8). The key point is that one-point extensions of homotopical epimorphisms are again homotopical epimorphisms in a way that we control the corresponding essential images of the restriction functors (see Theorem 8.8). This allows us to inductively take into account the rest of the representations.
- (iv) Finally, we mimic the cokernel construction by adding the cofiber of the map from the source of the original quiver to the object supporting the biproduct. Observe that this final category \tilde{Q} contains both the original quiver Q and the reflected quiver Q' as subcategories (see Figure 1) – this was the point

of passing to invertible n -cubes. Moreover, performing dual constructions starting from either side one obtains the same representations of \tilde{Q} . This is carried out in §9, and shows that Q and Q' are strongly stably equivalent.

6. HOMOTOPICAL EPIMORPHISMS

In the theory of derivators one constantly uses the fact that Kan extensions along fully faithful functors are again fully faithful (Examples 2.7). The following definition captures the case of fully faithful restriction functors.

Definition 6.1. Let $A, B \in \mathcal{Cat}$. A functor $u: A \rightarrow B$ is a **homotopical epimorphism** if the commutative square

$$(6.2) \quad \begin{array}{ccc} A & \xrightarrow{u} & B \\ u \downarrow & & \downarrow = \\ B & \xrightarrow{=} & B \end{array}$$

is homotopy exact.

Thus, a functor $u: A \rightarrow B$ is a homotopical epimorphism if and only if the counit $\epsilon: u_! u^* \rightarrow \text{id}$ is a natural isomorphism, i.e., if and only if the restriction functor $u^*: \mathcal{D}(B) \rightarrow \mathcal{D}(A)$ is fully faithful for any derivator \mathcal{D} . Moreover, an object $X \in \mathcal{D}(A)$ lies in the essential image of u^* if and only if the unit $\eta: \text{id} \rightarrow u^* u_!$ is an isomorphism on X . Using right Kan extensions instead, u is a homotopical epimorphism if and only if $\eta: \text{id} \rightarrow u_* u^*$ is an isomorphism. In this case X lies in the essential image of u^* if and only if $\epsilon: u^* u_* \rightarrow \text{id}$ is an isomorphism on X .

Remark 6.3. (i) The definition of a homotopical epimorphism seems to depend on the notion of a derivator, but this is not the case. It is immediate from [GPS13, Theorem 3.16] (which in turn relies essentially on work of Heller [Hel88] and Cisinski [Cis06]) that $u: A \rightarrow B$ is a homotopical epimorphism if and only if the functor

$$u^*: \text{Ho}(\mathbf{sSet}^B) \rightarrow \text{Ho}(\mathbf{sSet}^A)$$

is fully faithful. Here \mathbf{sSet} is the category of simplicial sets, and the simplicial presheaf categories are endowed with projective Kan–Quillen model structures [BK72]. Thus the notion ‘homotopical epimorphism’ only depends on the classical homotopy theory of spaces.

- (ii) In principle, one could consider the more general definition of a \mathcal{D} -homotopical epimorphism, i.e., a functor $u: A \rightarrow B$ such that $u^*: \mathcal{D}(B) \rightarrow \mathcal{D}(A)$ is fully faithful only for a specific derivator (or similarly for a specific class of derivators). However, in this paper we will have no use for this more general concept.

Remark 6.4. The motivation for this terminology comes from the following relation to *homological epimorphisms* in homological algebra. To start with, a *ring epimorphism* $u: A \rightarrow B$ is simply an epimorphism in the category of rings. Ring epimorphisms are characterized by the property that the forgetful functor $u^*: \text{Mod}(B) \rightarrow \text{Mod}(A)$ is fully faithful; see [Sil67, Sto73, GdlP87]. A **homological epimorphism** of rings is a homological analogue of the concept and has been first studied by Geigle and Lenzing [GL91, §4]. It is by definition a ring homomorphism $u: A \rightarrow B$ such

that the induced forgetful functor $u^*: D(B) \rightarrow D(A)$ is fully faithful. Recently, Pauksztello [Pau09] and Nicolas and Saorín [NS09, §4] studied the situation when a morphism $u: A \rightarrow B$ of differential-graded algebras or small differential-graded categories induces a fully faithful functor $u^*: D(B) \rightarrow D(A)$ between the derived categories. They again called the corresponding notion a homological epimorphism of dg-algebras or dg-categories, respectively, and showed that it is intimately related to smashing localizations of derived categories. In view of the previous remark, our concept of homotopical epimorphism is a direct generalization of homological epimorphisms to the simplicial context.

Let us collect a few examples and closure properties of homotopical epimorphisms.

- Proposition 6.5.** *(i) Homotopical epimorphisms are stable under compositions and contain the identities. The opposite of a homotopical epimorphism is again a homotopical epimorphism. If u is naturally isomorphic to v , then u is a homotopical epimorphism if and only if v is.*
- (ii) Equivalences of categories are homotopical epimorphisms. More generally, a functor admitting a fully faithful left adjoint (i.e., a coreflective colocalization) or a fully faithful right adjoint (i.e., a reflective localization) is a homotopical epimorphism.*
- (iii) Disjoint unions and finite products of homotopical epimorphisms are again homotopical epimorphisms.*

Proof. The statements in (i) are immediate. As for (ii), let $u: A \rightarrow B$ have a fully faithful left adjoint $v: B \rightarrow A$ and let $\eta: \text{id} \rightarrow uv$ and $\epsilon: vu \rightarrow \text{id}$ be the unit and the counit. The fully faithfulness of v is equivalent to η being an isomorphism. For every derivator \mathcal{D} , we obtain an adjunction $(u^*, v^*): \mathcal{D}^B \rightleftarrows \mathcal{D}^A$ with unit $\eta^*: \text{id} \rightarrow v^*u^*$ and counit $\epsilon^*: u^*v^* \rightarrow \text{id}$. Thus, the left adjoint u^* is fully faithful and this establishes the claims in (ii). A coproduct of homotopical epimorphisms is again a homotopical epimorphism by (Der1). Finally, note that the restriction functor along $u_1 \times u_2: A_1 \times A_2 \rightarrow B_1 \times B_2$ factors as

$$\mathcal{D}^{B_1 \times B_2} \cong (\mathcal{D}^{B_1})^{B_2} \xrightarrow{u_2^*} (\mathcal{D}^{B_1})^{A_2} \cong (\mathcal{D}^{A_2})^{B_1} \xrightarrow{u_1^*} (\mathcal{D}^{A_2})^{A_1} \cong \mathcal{D}^{A_1 \times A_2}.$$

Thus, if u_1, u_2 both are homotopical epimorphisms then the two restriction functors are fully faithful and $u_1 \times u_2$ is also a homotopical epimorphism. \square

We will need a combinatorial detection criterion for homotopical epimorphisms which allows us to establish examples which are not covered by Proposition 6.5. This is obtained by specializing a more general detection criterion for homotopy exact squares from [GPS13], which we recall next. Thus, let us again consider a natural transformation

$$(6.6) \quad \begin{array}{ccc} D & \xrightarrow{p} & A \\ q \downarrow & \searrow_{\alpha} & \downarrow u \\ B & \xrightarrow{v} & C \end{array}$$

living in a square of small categories.

Definition 6.7. For $a \in A$, $b \in B$, and $\gamma: u(a) \rightarrow v(b)$, let $(a/D/b)_\gamma$ be the category consisting of triples $(d \in D, a \xrightarrow{\phi} p(d), q(d) \xrightarrow{\psi} b)$ such that

$$\gamma = v\psi \circ \alpha_d \circ u\phi: u(a) \rightarrow up(d) \rightarrow vq(d) \rightarrow v(b).$$

Morphisms are morphism in D making the obvious triangles in A and B commute.

The category $(a/D/b)_\gamma$ consists of certain two-sided factorizations of γ through components of the natural transformation α . We will have a use for this category both in this generality but also under the simplifying assumptions that α, u , and v are identities.

Recall that associated to a small category E there is the simplicial set $N(E)$, the *nerve* of E , which in degree n is the set of strings of n composable morphisms in E .

Theorem 6.8 ([GPS13, Theorem 3.16]). *The square (6.6) is homotopy exact if and only if the nerve of $(a/D/b)_\gamma$ is weakly contractible for all a, b , and γ .*

Note that the converse direction of this theorem implies that the notion of homotopy exact squares depends only on the classical homotopy theory of spaces (see [GPS13, §3] for more details). For later reference, we now specialize this combinatorial criterion to the context of homotopical epimorphisms (see (6.2)).

Definition 6.9. Let $u: A \rightarrow B$ be a functor and $\gamma: b_1 \rightarrow b_2$ be a morphism in B . Then $(b_1/A/b_2)_\gamma$ is the category of factorizations of γ as $b_1 \rightarrow u(a) \rightarrow b_2$. A morphism $(a, b_1 \rightarrow u(a) \rightarrow b_2) \rightarrow (a', b_1 \rightarrow u(a') \rightarrow b_2)$ is a morphism $a \rightarrow a'$ in A making the obvious diagram in B commute.

Theorem 6.10. *A functor $u: A \rightarrow B$ is a homotopical epimorphism if and only if the categories $(b_1/A/b_2)_\gamma$ have weakly contractible nerves for all $b_1, b_2 \in B$ and all $\gamma: b_1 \rightarrow b_2$. If this is the case, then $u^*: \mathcal{D}^B \rightarrow \mathcal{D}^A$ induces an equivalence onto its essential image which consists precisely of the objects $X \in \mathcal{D}^A$ such that the adjunction unit $\eta: X \rightarrow u^*u_!(X)$ or equivalently the adjunction counit $\epsilon: u^*u_*(X) \rightarrow X$ is an isomorphism.*

Proof. The first part is immediate from Theorem 6.8. The characterization of the essential image of the restriction functor is immediate from the fact that there are adjunctions $(u_!, u^*)$ and (u^*, u_*) with u^* being fully faithful. \square

Corollary 6.11. *The projection functor $\pi_A: A \rightarrow \mathbb{1}$ is a homotopical epimorphism if and only if the nerve $N(A)$ is weakly contractible.*

Proof. It suffices to observe that the category $(*/A/*)_{{\rm id}}$ is isomorphic to A . Since this is the only case to be considered the result is immediate. \square

Thus, if $N(A)$ is weakly contractible, then any morphism $\pi_A^*X \rightarrow \pi_A^*Y$ can be written as $\pi_A^*(f)$ for a unique map $f: X \rightarrow Y$ in the underlying category $\mathcal{D}(\mathbb{1})$.

Remark 6.12. Recall that a functor $u: A \rightarrow B$ is a **homotopy equivalence** if the canonical mate

$$(\pi_A)!(\pi_A)^* \cong (\pi_B)u_!u^*(\pi_B)^* \rightarrow (\pi_B)!(\pi_B)^*$$

is an isomorphism in any derivator. Intuitively, a functor $A \rightarrow B$ is a homotopy equivalence if and only if it tells us that homotopy colimits of A -shaped constant diagrams and B -shaped constant diagrams are canonically isomorphic. Heller [Hel88]

and Cisinski [Cis06] showed that a functor $u: A \rightarrow B$ is a homotopy equivalence if and only if $N(u): N(A) \rightarrow N(B)$ is a weak homotopy equivalence. Since vertical pastings of homotopy exact squares are again homotopy exact, the diagram

$$\begin{array}{ccccc}
 A & \xrightarrow{u} & B & \xrightarrow{\pi_B} & \mathbb{1} \\
 u \downarrow & & \downarrow = & & \\
 B & \xrightarrow{=} & B & & \\
 \pi_B \downarrow & & \downarrow \pi_B & & \\
 \mathbb{1} & \xrightarrow{=} & \mathbb{1} & &
 \end{array}$$

tells us that homotopical epimorphisms are homotopy final functors. Recall from Examples 2.7 that these functors tell us something about homotopy colimits of not necessarily constant diagrams. In particular, by only considering constant diagrams we see that homotopy final functors are homotopy equivalences. As a summary, there are the following relations between these various notions:

$$\begin{aligned}
 & u: A \rightarrow B \text{ is a homotopical epimorphism} \\
 \Rightarrow & u: A \rightarrow B \text{ is homotopy final} \\
 \Rightarrow & u: A \rightarrow B \text{ is a homotopy equivalence} \\
 \Leftrightarrow & N(u): N(A) \rightarrow N(B) \text{ is a weak homotopy equivalence}
 \end{aligned}$$

The notion ‘homotopical epimorphism’ certainly deserves to be studied more systematically. Here however we only develop what is necessary for our approach to abstract reflection functors: in §7 we establish a key example which allows us to give a more symmetric description of biproduct diagrams and in §8 we study the behavior of homotopical epimorphisms with respect to one-point extensions.

7. FINITE BIPRODUCTS VIA INVERTIBLE n -CUBES

The aim of this section is to show that the ‘passage from the n -cube to the invertible n -cube’ is a homotopical epimorphism, and to understand the essential image of the associated restriction functor (see Corollary 7.5). To formalize this, we begin by the following definition.

Definition 7.1. Let R be the category obtained from $[2] = (0 \rightarrow 1 \rightarrow 2)$ by freely inverting the morphism $0 \rightarrow 2$. The localization functor is denoted $p: [2] \rightarrow R$.

Thus, the category R has objects 0, 1, and 2 and both objects 0 and 2 are zero objects. Moreover, the object $1 \in R$ has a non-identity endomorphism $t: 1 \rightarrow 1$ which is given by the zero map and hence is idempotent. A picture of the category suppressing the identity morphisms is

$$0 \begin{array}{c} \xrightarrow{\quad} \\ \xleftarrow{\quad} \end{array} 1 \begin{array}{c} \xrightarrow{\quad} \\ \xleftarrow{\quad} \end{array} 2$$

in which the unlabeled loop is the idempotent $t: 1 \rightarrow 1$ and in which the maps $0 \rightarrow 2$ and $2 \rightarrow 0$ are inverse isomorphisms. Thus, in total there are ten morphisms in R .

Since the following proposition is essential to our approach to reflection functors in §9 and since this is the first instance of a proof in this paper in which homotopy exact squares and the calculus of mates (see (2.5) and (2.6)) play a key role, we include a rather detailed proof. Note that the result is not formal in that p is

neither a reflective localization nor a coreflective colocalization, and the result is hence not covered by [Proposition 6.5](#).

Proposition 7.2. *The localization functor $p: [2] \rightarrow R$ is a homotopical epimorphism. For every derivator \mathcal{D} , $p^*: \mathcal{D}^R \rightarrow \mathcal{D}^{[2]}$ induces an equivalence onto the full subderivator of $\mathcal{D}^{[2]}$ spanned by the objects X such that $X_0 \rightarrow X_2$ is an isomorphism.*

Proof. In order to check that p is a homotopical epimorphism, we apply the combinatorial detection criterion [Theorem 6.10](#). In our situation this amounts to showing that the ten different categories $(r_1/[2]/r_2)_\gamma$ have weakly contractible nerves. Let us content ourselves by giving the details for the case of the non-identity idempotent $\gamma = t: 1 \rightarrow 1$. In this case, $(1/[2]/1)_t$ is freely generated by the graph

$$\begin{array}{ccccc}
 & (0, 1 \rightarrow 0 \rightarrow 1) & & & \\
 & \downarrow (0 \rightarrow 1) & \searrow (0 \rightarrow 1) & & \\
 (1, 1 \xrightarrow{=} 1 \xrightarrow{t} 1) & (1, 1 \xrightarrow{t} 1 \xrightarrow{t} 1) & & (1, 1 \xrightarrow{t} 1 \xrightarrow{=} 1) & \\
 & \downarrow (1 \rightarrow 2) & & & \\
 & (2, 1 \rightarrow 2 \rightarrow 1) & & &
 \end{array}$$

(Note: The diagram above is a simplified representation of the graph structure shown in the image. The original image shows a more complex graph with additional arrows and labels like (1→2) and (0→1) connecting the nodes.)

where the undecorated morphisms are the unique maps in $[2]$ and R , respectively. Thus the nerve of this category is weakly contractible. The remaining nine cases are similar: one checks that the nerves of the respective categories are certain very small oriented trees (with at most three edges in the remaining cases) and that the nerves are hence also weakly contractible. Thus, [Theorem 6.10](#) implies that $p: [2] \rightarrow R$ is a homotopical epimorphism.

Let us now describe the essential image of p^* . As the morphism $0 \rightarrow 2$ is invertible in R it is immediate that $X_0 \rightarrow X_2$ is an isomorphism for all X in the essential image of p^* . The converse is more involved. Since p is a homotopical epimorphism, we are given an adjunction $(p_!, p^*)$ with a fully faithful right adjoint. Thus, X lies in the essential image of p^* if and only if the adjunction unit $\eta: X \rightarrow p^*p_!(X)$ is an isomorphism. Note that the adjunction unit is the canonical mate [\(2.5\)](#) associated to the square on the right in

$$\begin{array}{ccccc}
 (\mathrm{id}/i) & \longrightarrow & [2] & \xrightarrow{=} & [2] \\
 \downarrow & \not\cong & \downarrow & \not\cong_{\mathrm{id}} & \downarrow p \\
 \mathbb{1} & \xrightarrow{i} & [2] & \xrightarrow{p} & R.
 \end{array}$$

By (Der2), isomorphisms are detected pointwise and it is hence enough to show that the mate is an isomorphism for each $i = 0, 1, 2$. Given $i = 0, 1, 2$ one uses the compatibility of mates with pasting (see [Examples 2.7](#)) together with (Der4) to conclude that η_i is an isomorphism at X if and only if the canonical mate of the above pasting is an isomorphism at X . Since there is an obvious isomorphism $(\mathrm{id}/i) \cong [i]$, we can use the homotopy final functor $\mathbb{1} \xrightarrow{i} [i]$ classifying the terminal object (see [Examples 2.7](#)) and again the compatibility of mates with pasting to

conclude that η_i is an isomorphism at X if and only if the mate associated to

$$(7.3) \quad \begin{array}{ccc} \mathbb{1} & \xrightarrow{i} & [2] \\ \downarrow & \not\cong_{\text{id}} & \downarrow p \\ \mathbb{1} & \xrightarrow{i} & R \end{array}$$

is an isomorphism at X .

We now re-express this statement for $i = 0, 1, 2$ showing that it is satisfied under the additional assumption that $X_0 \rightarrow X_2$ is an isomorphism. We begin with the trickiest case which is the case of $i = 1$, and start by considering the homotopy exact square

$$\begin{array}{ccc} (p/1) & \longrightarrow & [2] \\ \downarrow & \not\cong & \downarrow p \\ \mathbb{1} & \xrightarrow{1} & R \end{array}$$

given by (Der4). The category $(p/1)$ is freely generated by the graph

$$(7.4) \quad \begin{array}{c} (0, 0 \rightarrow 1) \xrightarrow{(0 \rightarrow 1)} (1, 1 \xrightarrow{t} 1) \xrightarrow{(1 \rightarrow 2)} (2, 2 \rightarrow 1) \\ (0 \rightarrow 1) \downarrow \\ (1, 1 \xrightarrow{\overline{\tau}} 1) \end{array}$$

where again the undecorated morphisms are the unique maps in $[2]$ or R , respectively. The inclusion $\ulcorner \rightarrow (p/1)$ which does not hit the object $(1, 1 \xrightarrow{t} 1)$ is a right adjoint and hence homotopy final by [Examples 2.7](#). Using the compatibility of homotopy exact squares with respect to pasting again, we conclude that the pasting of the two squares to the right in

$$\begin{array}{ccccccc} \mathbb{1} & \xrightarrow{1} & [1] & \xrightarrow{j} & \ulcorner & \longrightarrow & (p/1) \longrightarrow [2] \\ \downarrow & & \downarrow & & \downarrow & & \downarrow p \\ \mathbb{1} & \longrightarrow & \mathbb{1} & \longrightarrow & \mathbb{1} & \longrightarrow & \mathbb{1} \xrightarrow{1} R \end{array}$$

is homotopy exact. In the above diagram $j: [1] \rightarrow \ulcorner$ is the vertical inclusion. Using the explicit description of the slice category (7.4) we see that the pasting agrees with (7.3) for $i = 1$. Since $1 \in [1]$ is a terminal object, the square to the very left is homotopy exact by [Examples 2.7](#). Thus, using once more the compatibility of mates with pasting, it remains to show the following: if $X_0 \rightarrow X_2$ is an isomorphism, then the mate associated to the second square from the left is an isomorphism on the restriction of X to a diagram Y of shape \ulcorner . By (7.4), the underlying diagram of this restriction Y is

$$\begin{array}{ccc} X_0 & \xrightarrow{\cong} & X_2 \\ \downarrow & & \\ X_1 & & \end{array}$$

and we hence expect its colimit to be isomorphic to X_1 (and in particular to only depend on the restriction along j). Formally, this follows by considering the factorization of the square as

$$\begin{array}{ccc} [1] & \xrightarrow{j} & \ulcorner \\ \downarrow & & \downarrow \\ \ulcorner & \longrightarrow & \ulcorner \\ \downarrow & & \downarrow \\ \mathbb{1} & \longrightarrow & \mathbb{1} \end{array}$$

and by observing that the bottom square is trivially homotopy exact. Thus the mate associated to this diagram is an isomorphism on Y if and only if the adjunction counit $j_! j^*(Y) \rightarrow Y$ is an isomorphism. Since j is fully faithful, it is enough to check this at the upper right corner $(1, 0)$ ([Gro13, Lemma 1.21]). But using (Der4) this is the case if and only if $X_0 \rightarrow X_2$ is an isomorphism (see also the proof of [Gro13, Proposition 3.12]). Thus, we conclude that η_1 is an isomorphism for all X such that $X_0 \rightarrow X_2$ are isomorphisms.

The remaining two cases are much simpler. To show that η_2 is an isomorphism it suffices to use a finality argument (see Examples 2.7) as expressed by the pasting situation

$$\begin{array}{ccccccc} \mathbb{1} & \xrightarrow{2} & [2] & \xrightarrow{\cong} & (p/2) & \longrightarrow & [2] \\ \downarrow & & \downarrow & & \downarrow & \nearrow & \downarrow p \\ \mathbb{1} & \longrightarrow & \mathbb{1} & \longrightarrow & \mathbb{1} & \xrightarrow{2} & R. \end{array}$$

Since this pasting is the same as the square (7.3) for $i = 2$ we conclude that η_2 is an isomorphism without any further assumption on X . Finally, for $i = 0$ a finality argument tells us that the pasting

$$\begin{array}{ccccc} \mathbb{1} & \xrightarrow{2} & (p/0) & \longrightarrow & [2] \\ \downarrow & & \downarrow & \nearrow & \downarrow p \\ \mathbb{1} & \longrightarrow & \mathbb{1} & \xrightarrow{0} & R \end{array}$$

is homotopy exact, and the mate of the pasting is hence an isomorphism for an arbitrary X . Note that this is not yet the diagram (7.3) used to calculate η_0 but that (7.3) is obtained from it by a vertical pasting as in

$$\begin{array}{ccc} \mathbb{1} & \xrightarrow{0} & [2] \\ \downarrow & \nearrow & \downarrow \\ \mathbb{1} & \xrightarrow{2} & [2] \\ \downarrow & \nearrow & \downarrow p \\ \mathbb{1} & \xrightarrow{0} & R. \end{array}$$

Now the fact that $X_0 \rightarrow X_2$ is an isomorphism says precisely that the mate associated to the top square is an isomorphism when applied to X . Thus, also η_0 is an isomorphism, and this concludes the proof. \square

There is the following variant of the proposition for finitely many coordinates.

Corollary 7.5. *The functor $q = p^{\times n}: [2]^{\times n} \rightarrow R^{\times n}$ is a homotopical epimorphism and $q^*: \mathcal{D}^{R^{\times n}} \rightarrow \mathcal{D}^{[2]^{\times n}}$ induces an equivalence onto the full subderivator of $\mathcal{D}^{[2]^{\times n}}$ spanned by all diagrams X such that*

$$X_{i_1, \dots, i_{k-1}, 0, i_{k+1}, \dots, i_n} \rightarrow X_{i_1, \dots, i_{k-1}, 2, i_{k+1}, \dots, i_n}$$

is an isomorphism for all $i_1, \dots, i_{k-1}, i_{k+1}, \dots, i_n$ and k .

Proof. The fact that q is a homotopical epimorphism follows immediately from [Proposition 7.2](#) and the stability of homotopical epimorphisms under finite products (see [Proposition 6.5](#)). The essential image of q^* can be described inductively using [Proposition 7.2](#) again. \square

It is immediate to see that this equivalence can be restricted to equivalences of full subderivators spanned by diagrams satisfying certain vanishing or exactness conditions. Let \mathcal{D} be a stable derivator and let $\mathcal{D}^{R^n, \text{ex}}$ be the full subprederivator of \mathcal{D}^{R^n} spanned by all diagrams such that

- (i) all subcubes are strongly bicartesian and
- (ii) the values at all corners are trivial.

Corollary 7.6. *Let \mathcal{D} be a stable derivator. For every $n \geq 2$ there is an equivalence $\mathcal{D} \times \dots \times \mathcal{D} \simeq \mathcal{D}^{n, 1} \rightarrow \mathcal{D}^{R^n, \text{ex}}$ which is natural with respect to exact morphisms.*

Proof. By axiom (Der1) there is a natural equivalence $\mathcal{D} \times \dots \times \mathcal{D} \simeq \mathcal{D}^{n, 1}$. By [Proposition 4.9](#) there is a natural equivalence $\mathcal{D}^{n, 1} \simeq \mathcal{D}^{[2]^n, \text{ex}}$. The explicit descriptions of $\mathcal{D}^{[2]^n, \text{ex}}$ and $\mathcal{D}^{R^n, \text{ex}}$ together with [Corollary 7.5](#) imply that we also have a natural equivalence between these two derivators which concludes the proof. \square

Thus, given a stable derivator, this corollary makes precise a way to encode coherent versions of finite biproduct diagrams in a completely symmetric fashion. Recall from [§5](#) that this is roughly step (ii) in our strategy for the construction of reflection functors in [§9](#).

8. ONE-POINT EXTENSIONS

Given an arbitrary tree Q and a source $q_0 \in Q$ of valence n , we would like to glue an n -cube $[2]^n$ into the quiver, invert the cube, and then show that this localization functor is a homotopical epimorphism. Moreover, we want to show that the restriction functor induces an equivalence onto the derivator of all diagrams inverting the length two morphisms. Instead of trying to show this directly, we proceed differently. The following simple formalism of one-point extensions of small categories allows us to inductively reduce this more general context to the situation we already studied in [§7](#).

Definition 8.1. Let A and A' be small categories. The category A' is a **one-point extension** of A if there is a pushout diagram

$$(8.2) \quad \begin{array}{ccc} \mathbb{1} & \xrightarrow{a} & A \\ \downarrow i & & \downarrow j_A \\ [1] & \longrightarrow & A'. \end{array}$$

Thus, we simply attach a morphism to A by identifying either its source or its target with a given object in A . Note that A' comes with two full subcategories $A, A' - A \cong \mathbb{1}$ which are *convex* in the sense that any morphism in these subcategories cannot factor through an object in the complement.

If A' is a one-point extension of A as in (8.2) and if $u: A \rightarrow B$ is a functor, then there is a canonical functor $u': A' \rightarrow B'$, where B' is the corresponding one-point extension of B at $u(a)$. We say that $u': A' \rightarrow B'$ is a **one-point extension** of $u: A \rightarrow B$, and in such a situation there is a commutative diagram

$$(8.3) \quad \begin{array}{ccc} A & \xrightarrow{j_A} & A' \\ \downarrow u & & \downarrow u' \\ B & \xrightarrow{j_B} & B'. \end{array}$$

Remark 8.4. Our terminology is again based on a similar construction common in representation theory; see [ARS97, §III.2]. If $A = kQ/I$ is a finite dimensional path algebra and $M \in \text{Mod}(A)$, a **one-point extension** is the triangular matrix ring $B = \begin{pmatrix} k & 0 \\ M & A \end{pmatrix}$ with the obvious matrix multiplication. It is easy to see that B is of the form $B = kQ'/I'$, where Q' is obtained from Q by adding a new source and arrows from it to vertices of Q . If M is an indecomposable projective A -module, then Q' is obtained from Q by adding precisely one arrow from the source and no new relations. Thus, this is precisely a linearized version of our one-point extension defined above.

Lemma 8.5. *Let $u': A' \rightarrow B'$ be a one-point extension of $u: A \rightarrow B$.*

- (i) *If A' is obtained by attaching a morphism to its target, then $j_A: A \rightarrow A'$ admits a left adjoint l_A , i.e., A is a reflective localization of A' . The functors l_A and l_B can be chosen such that $u \circ l_A = l_B \circ u': A' \rightarrow B$.*
- (ii) *If A' is obtained by attaching a morphism to its source, then $j_A: A \rightarrow A'$ admits a right adjoint r_A , i.e., A is a coreflective colocalization of A' . The functors r_A and r_B can be chosen such that $u \circ r_A = r_B \circ u': A' \rightarrow B$.*

In particular, the nerves $N(A)$ and $N(A')$ are homotopy equivalent.

Proof. We consider the case in (i). The left adjoint l_A can be chosen to be the identity on $A \subseteq A'$. If the new morphism in A' is given by $f: a_{-1} \rightarrow a_0$ with $a_0 \in A$, then we set $l_A(a_{-1}) = a_0$. Moreover, any map $\gamma: a_{-1} \rightarrow a$ with $a \in A$ factors uniquely as $a_{-1} \xrightarrow{f} a_0 \xrightarrow{\tilde{\gamma}} a$ and we set $l_A(\gamma) = \tilde{\gamma}$. One checks that this defines the desired left adjoint to the fully faithful $j_A: A \rightarrow A'$, and the relation $u \circ l_A = l_B \circ u'$ is immediate. Since natural transformations induce simplicial homotopies between the induced maps on nerves it is immediate that any adjunction gives rise to a simplicial homotopy equivalence. \square

Proposition 8.6. *A one-point extension $u': A' \rightarrow B'$ of a homotopical epimorphism $u: A \rightarrow B$ is a homotopical epimorphism.*

Proof. We establish this result by applying the combinatorial detection principle for homotopical epimorphisms (Theorem 6.10). Let us assume that A' is obtained from A by adding an object a_{-1} together with a morphism $f: a_{-1} \rightarrow a_0$ (the case of the other orientation is similar). Then in B' there is the corresponding morphism $u'(f): b_{-1} \rightarrow b_0 = u(a_0)$ describing B' as a one-point extension of B . We note that the full subcategories $\{b_{-1}\}, B \subseteq B'$ are convex. By Theorem 6.10 we have to show that the categories $(b_1/A'/b_2)_\gamma$ have weakly contractible nerves for all morphisms $\gamma: b_1 \rightarrow b_2$ in B' . There are the following four cases to be considered.

- (i) Let $b_1, b_2 \in B \subseteq B'$. Then the convexity of B gives us an isomorphism of categories $(b_1/A'/b_2)_\gamma \cong (b_1/A/b_2)_\gamma$ where $(b_1/A/b_2)_\gamma$ is defined using the functor $u: A \rightarrow B$. But since $u: A \rightarrow B$ is a homotopical epimorphism this latter category has a weakly contractible nerve.
- (ii) If $b_1 = b_2 = b_{-1}$ then there is only the identity map $\gamma = \text{id}$ and it is immediate that $(b_{-1}/A'/b_{-1})_{\text{id}} \cong \mathbb{1}$ so that the nerve is weakly contractible.
- (iii) If $b_1 \in B$ and $b_2 = b_{-1}$ then there is no map γ to be considered.
- (iv) The remaining case is given by $b_1 = b_{-1}$ and $b_2 \in B$. By definition of B' any such $\gamma: b_{-1} \rightarrow b_2$ can be uniquely factored as $u'(f): b_{-1} \rightarrow b_0$ followed by a map $\tilde{\gamma}: b_0 \rightarrow b_2$ in B . We leave it to the reader to check that this implies that the category $(b_{-1}/A'/b_2)_\gamma$ is obtained by a one-point extension from $(b_0/A/b_2)_{\tilde{\gamma}}$ which in turn is defined using the functor $u: A \rightarrow B$. The new object is given by $(a_{-1}, \text{id}: b_{-1} \rightarrow b_{-1}, \gamma: b_{-1} \rightarrow b_2)$, and it is attached to $(a_0, \text{id}: b_0 \rightarrow b_0, \tilde{\gamma}: b_0 \rightarrow b_2)$ by means of the map f . But since $u: A \rightarrow B$ is a homotopical epimorphism, $(b_0/A/b_2)_{\tilde{\gamma}}$ has a weakly contractible nerve and so the same is true for $(b_{-1}/A'/b_2)_\gamma$ by Lemma 8.5.

Thus in all cases the categories $(b_1/A'/b_2)_\gamma$ have weakly contractible nerves, and we are done by Theorem 6.10. \square

The next aim is to describe the essential image of u'^* in terms of the one of u^* and for that purpose we show that the square (8.3) is homotopy exact. We allow ourselves to re-emphasize that, in general, one has to be careful about what one means by saying that a *commutative square* is homotopy exact. (Recall that there is such a potential source of confusion in the case of pullback squares involving Grothendieck (op)fibrations (see [Gro13, §1.3])). Even at the risk of being picky, let us consider the following squares

$$\begin{array}{ccc} A & \xrightarrow{j_A} & A' \\ u \downarrow & \swarrow \text{id} & \downarrow u' \\ B & \xrightarrow{j_B} & B' \end{array} \quad \begin{array}{ccc} A & \xrightarrow{j_A} & A' \\ u \downarrow & \swarrow \text{id} & \downarrow u' \\ B & \xrightarrow{j_B} & B' \end{array}$$

and make the following case distinction.

Proposition 8.7. *Let $u': A' \rightarrow B'$ be a one-point extension of $u: A \rightarrow B$.*

- (i) *The above square on the left is homotopy exact, i.e., the canonical mate $u_! j_A^* \rightarrow j_B^* u'_!$ or equivalently the canonical mate $(u')^*(j_B)_* \rightarrow (j_A)_* u^*$ is an isomorphism.*

- (ii) *The above square on the right is homotopy exact, i.e., the canonical mate $j_B^* u'_* \rightarrow u_* j_A^*$ or equivalently the canonical mate $(j_A)_! u^* \rightarrow (u')^* (j_B)_!$ is an isomorphism.*

Proof. We take care of the case that A' is obtained from A by attaching a morphism $f: a_{-1} \rightarrow j_A(a_0)$ for some object $a_0 \in A$, and hence similarly for B' with the morphism $u'(f): b_{-1} \rightarrow j_B(b_0)$. We want to prove the results by an application of [Theorem 6.8](#), and we begin with statement (i). Thus, for $a' \in A', b \in B$, and $\gamma: u'(a') \rightarrow j_B(b)$ we have to show that the category $(a'/A/b)_\gamma$ has a weakly contractible nerve. We recall that objects are triples $(a \in A, a' \rightarrow j_A(a), u(a) \rightarrow b)$ such that the composition

$$u'(a') \rightarrow u'j_A(a) \xrightarrow{\cong} j_B u(a) \rightarrow j_B(b)$$

coincides with γ . Morphisms are morphisms in A making the obvious triangles in A' and B commute. There are the following two cases.

- (a) If a' is different from a_{-1} , then $a' = j_A(\tilde{a})$ for a unique object \tilde{a} in A and $\gamma: u'(a') = j_B u(\tilde{a}) \rightarrow j_B(b)$ can be uniquely written as $j_B(\tilde{\gamma})$. We claim that $(\tilde{a}, \text{id}: a' = j_A(\tilde{a}) \rightarrow j_A(\tilde{a}), \tilde{\gamma}: u(\tilde{a}) \rightarrow b)$ is an initial object in $(a'/A/b)_\gamma$. In fact, let $(a, j_A(g): a' = j_A(\tilde{a}) \rightarrow j_A(a), h: u(a) \rightarrow b)$ be a further object in $(a'/A/b)_\gamma$. Then we have to show that there is a unique map $\tilde{a} \rightarrow a$ in A such that

$$j_A(g): a' \xrightarrow{\cong} j_A(\tilde{a}) \rightarrow j_A(a) \quad \text{and} \quad \tilde{\gamma}: u(\tilde{a}) \rightarrow u(a) \xrightarrow{h} b.$$

Clearly, the first condition forces us to choose $g: \tilde{a} \rightarrow a$ and this map also satisfies the second condition since $(a, j_A(g), h)$ is an object of $(a'/A/b)_\gamma$. Thus, the nerve of $(a'/A/b)_\gamma$ is weakly contractible.

- (b) It remains to consider the case that $a' = a_{-1} \in A'$. Thus we are given a map $\gamma: u'(a_{-1}) = b_{-1} \rightarrow j_B(b)$ which hence factors uniquely as

$$\gamma = j_B(\tilde{\gamma}) \circ u'(f): u'(a_{-1}) \rightarrow u'(j_A(a_0)) \xrightarrow{\cong} j_B(b_0) \rightarrow j_B(b)$$

for a unique map $\tilde{\gamma}: b_0 \rightarrow b$ in B . Thus, the triple $(a_0, f, \tilde{\gamma})$ defines an object in $(a_{-1}/A/b)_\gamma$ and we claim that it is initial. In fact, given an object $(a \in A, g: a_{-1} \rightarrow j_A(a), h: u(a) \rightarrow b)$ in $(a_{-1}/A/b)_\gamma$, then g factors uniquely as $g = j_A(g') \circ f: a_{-1} \rightarrow j_A(a_0) \rightarrow j_A(a)$ for a unique $g': a_0 \rightarrow a$ in A . This implies that there is at most one morphism $(a_0, f, \tilde{\gamma}) \rightarrow (a, g, h)$ necessarily given by g' . It remains to check that $\tilde{\gamma}$ can be written as $h \circ u(g')$ which follows immediately from the fully faithfulness of j_B and because (a, g, h) lies in $(a_{-1}/A/b)_\gamma$. Thus, also in this case the nerve is weakly contractible.

Thus, [Theorem 6.8](#) concludes the proof of statement (i). We also use this theorem to establish (ii), and we thus have to show that the category $(b/A/a')_\gamma$ has a weakly contractible nerve for $b \in B, a' \in A'$, and $\gamma: j_B(b) \rightarrow u'(a')$. If $a' = a_{-1}$ then there is no such map γ to be considered, and we can thus assume that $a' = j_A(a)$ and $\gamma = j_B(\tilde{\gamma}): j_B(b) \rightarrow u'j_A(a) = j_B u(a)$. But $(a, \tilde{\gamma}: b \rightarrow u(a), \text{id}: j_A(a) \rightarrow j_A(a))$ defines a terminal object in $(b/A/j_A(a))_{j_B(\tilde{\gamma})}$ because every object in this category can be uniquely written as $(a_1, g: b \rightarrow u(a), j_A(h): j_A(a_1) \rightarrow j_A(a))$ and the map $h: a_1 \rightarrow a$ defines the unique morphism $(a_1, g, j_A(h)) \rightarrow (a, \tilde{\gamma}, \text{id})$. Thus, a further application of [Theorem 6.8](#) establishes (ii). \square

We just showed that the square (8.3) is homotopy exact, independently of the orientation of the identity transformation and also independently of the orientation of the morphism attached in the one-point extensions.

Theorem 8.8. *Let \mathcal{D} be a derivator and let $u': A' \rightarrow B'$ be a one-point extension of a functor $u: A \rightarrow B$ in \mathcal{Cat} ,*

$$\begin{array}{ccc} A & \xrightarrow{j_A} & A' \\ u \downarrow & & \downarrow u' \\ B & \xrightarrow{j_B} & B'. \end{array}$$

- (i) *If u is a homotopical epimorphism, so is u' and $X' \in \mathcal{D}^{A'}$ lies in the essential image of $u'_*: \mathcal{D}^{B'} \rightarrow \mathcal{D}^{A'}$ if and only if $j_A^*(X')$ lies in the essential image of $u_*: \mathcal{D}^B \rightarrow \mathcal{D}^A$.*
- (ii) *If u is fully faithful, so is u' . Moreover, $Y' \in \mathcal{D}^{B'}$ lies in the essential image of $u'_*: \mathcal{D}^{A'} \rightarrow \mathcal{D}^{B'}$ if and only if $j_B^*(Y')$ lies in the essential image of $u_*: \mathcal{D}^A \rightarrow \mathcal{D}^B$, and Y' lies in the essential image of $u'_!: \mathcal{D}^{A'} \rightarrow \mathcal{D}^{B'}$ if and only if $j_B^*(Y')$ lies in the essential image of $u_!: \mathcal{D}^A \rightarrow \mathcal{D}^B$.*

Proof. (i) We first give the details for the case that A' is obtained from A by attaching a map $a_{-1} \rightarrow a_0$ with $a_0 \in A$, and then say a few words about the other case. Let $\eta: \text{id} \rightarrow u^*u_!$ and $\eta': \text{id} \rightarrow u'^*u'_!$ be the adjunction units. Since u is a homotopical epimorphism, the same is true for u' by Proposition 8.6. We thus have to show that η' is an isomorphism on $X' \in \mathcal{D}^{A'}$ if and only if η is an isomorphism on the restriction $j_A^*(X') \in \mathcal{D}^A$.

We begin by showing that η' is an isomorphism if and only if $j_A^*\eta'$ is an isomorphism, i.e., that η' is always an isomorphism when evaluated at a_{-1} . This follows easily from the two diagrams

$$\begin{array}{ccc} (\text{id}/a_{-1}) & \xrightarrow{p} & A' \xrightarrow{=} A' \\ \pi \downarrow & \not\cong & \downarrow u' \\ \mathbb{1} & \xrightarrow{a_{-1}} & A' \xrightarrow{u'} B', \end{array} \quad \begin{array}{ccc} (u'/u'(a_{-1})) & \longrightarrow & A' \\ \downarrow & \not\cong & \downarrow u' \\ \mathbb{1} & \xrightarrow{u'(a_{-1})} & B'. \end{array}$$

By the compatibility of mates with pasting, the canonical mate associated to the diagram on the left factors as

$$\pi_! p^* \xrightarrow{\cong} (a_{-1})^* \xrightarrow{\eta'_{a_{-1}}} (a_{-1})^*(u')^*(u')_!$$

of which the first map is an isomorphism by (Der4). Thus, this mate is an isomorphism if and only if $\eta'_{a_{-1}}$ is an isomorphism. Note that in both diagrams the upper left corner is a terminal category (for the case on the right recall that A' is obtained from A by attaching a new morphism to its target). It is then obvious that the two pastings can be identified and that the diagram on the right is homotopy exact by (Der4). Thus, $\eta'_{a_{-1}}$ is always an isomorphism as desired.

After this reduction step it suffices to consider the following factorizations

$$\begin{array}{ccc}
 A & \xrightarrow{j_A} & A' \xrightarrow{=} A' \\
 \downarrow & & \downarrow = \downarrow u' \\
 A & \xrightarrow{j_A} & A' \xrightarrow{u'} B'
 \end{array}
 =
 \begin{array}{ccc}
 A & \xrightarrow{=} & A \xrightarrow{j_A} A' \\
 \downarrow & & \downarrow u \not\llcorner_{\text{id}} \downarrow u' \\
 A & \xrightarrow{u} & B \xrightarrow{j_B} B'
 \end{array}$$

of the identity transformation. Since the square to the very right is homotopy exact (statement (i) of [Proposition 8.7](#)), we see that the mate $\beta: u_! j_A^* \rightarrow j_B^* u'_!$ is a natural isomorphism. Using the compatibility of mates with respect to pasting, we obtain that $j_A^* \eta'$ is an isomorphism if and only if $u^* \beta \cdot \eta j_A^*$ is an isomorphism if and only if ηj_A^* is an isomorphism. Together with the reduction step this implies that η' is an isomorphism if and only if ηj_A^* is an isomorphism, concluding the proof in the first case.

The case of one-point extensions obtained by attaching a new morphism to its source is dual in that one uses the adjunction *counits* to characterize the essential images of the respective restriction functors. The first reduction step is then the same while the second step is based on statement (ii) of [Proposition 8.7](#).

(ii) If u is fully faithful, then clearly also u' is fully faithful. Suppose now that Y' is in the essential image of $u'_*: \mathcal{D}^{A'} \rightarrow \mathcal{D}^{B'}$, or in other words that the adjunction unit $\eta': Y' \rightarrow u'_* u'^*(Y')$ is an isomorphism. By [[Gro13](#), Lemma 1.21(2)], this is equivalent to saying that $\eta'_{b'}: Y'_{b'} \rightarrow u'_* u'^*(Y')_{b'}$ is an isomorphism for each object $b' \in B' - u'(A')$. By construction, any such b' can be uniquely written as $j_B(b)$ with $b \in B - u(A)$. To re-express differently the condition that $\eta'_{b'}, b' = j_B(b)$, is an isomorphism, we consider the following pasting

$$(8.9) \quad \begin{array}{ccccc}
 (b/u) & \xrightarrow{j} & (b'/u') & \xrightarrow{p'} & A' \xrightarrow{u'} B' \\
 \downarrow & & \downarrow & \nearrow & \downarrow u' \downarrow = \\
 \mathbb{1} & \xrightarrow{\quad} & \mathbb{1} & \xrightarrow{b'} & B' \xrightarrow{=} B'
 \end{array}$$

in which the square in the middle is a comma square as in (Der4). The functor j is obtained by applying j_A to the object component and j_B to the morphism component, and it thus sends $(a, b \rightarrow u(a))$ to $(j_A(a), b' \rightarrow u'(j_A(a)))$. By functoriality with respect to pasting, the canonical mate associated to (8.9) applied to Y' factors as

$$Y'_{b'} \xrightarrow{\eta'} u'_* u'^*(Y')_{b'} \xrightarrow{\cong} \lim_{(b'/u')} p'^* u'^*(Y') \rightarrow \lim_{(b/u)} j^* p'^* u'^*(Y').$$

The first morphism here is precisely $\eta'_{b'}$ and the second one is an isomorphism by (Der4). To show that also the third morphism is an isomorphism we verify that $j: (b/u) \rightarrow (b'/u')$ is homotopy cofinal. There are the following two cases. Let A' be obtained from A by attaching a morphism $a_0 \rightarrow a_1$ with $a_0 \in A$. Recall from [Lemma 8.5\(ii\)](#) that $j_A: A \rightarrow A'$ and $j_B: B \rightarrow B'$ admit compatible right adjoints r_A and r_B . Together they define a functor $r: (b'/u') \rightarrow (b/u)$ by applying r_A to the object component and r_B to the morphism component. Thus, r sends $(a', b' \rightarrow u'(a'))$ to $(r_A(a'), b \rightarrow u(r_A(a')))$, and one checks that r is right adjoint to j . It is a consequence of [Examples 2.7](#) that j is homotopy cofinal. In the other case A' is obtained from A by attaching a morphism $a_{-1} \rightarrow a_0$ with $a_0 \in A$. Then it is immediate that $j: (b/u) \rightarrow (b'/u')$ is homotopy cofinal since in that case j is even

an isomorphism of categories (there are no morphisms from b' to $u'(a_{-1})$). Thus, as an upshot $Y' \in \mathcal{D}^{B'}$ lies in the essential image of u'_* if and only if $\eta'_{b'}, b' = j_B(b)$, is an isomorphism for all $b \in B - u(A)$ if and only if the canonical mate of the pasting (8.9) is an isomorphism on Y' for all $b \in B - u(A)$.

We note that the pasting (8.9) can be rewritten as

$$\begin{array}{ccccccc} (b/u) & \xrightarrow{p} & A & \xrightarrow{u} & B & \xrightarrow{j_B} & B' \\ \downarrow & \nearrow & \downarrow u & & \downarrow = & & \downarrow = \\ \mathbb{1} & \xrightarrow{b} & B & \xrightarrow{=} & B & \xrightarrow{j_B} & B'. \end{array}$$

The square on the right is trivially homotopy exact and the square on the left is homotopy exact by (Der4). Thus, the canonical mate associated to this pasting is an isomorphism on $Y' \in \mathcal{D}^{B'}$ if and only if $\eta_b: j_B^*(Y')_b \rightarrow (u_* u^* j_B^*(Y'))_b$ is an isomorphism. Appealing to [Gro13, Lemma 1.21(2)] again, this is the case for all $b \in B - u(A)$ if and only if $j_B^*(Y')$ is in the essential image of u_* . This shows that Y' lies in the essential image of u'_* if and only if $j_B^*(Y')$ lies in the essential image of u_* . The case of $u'_!$ is formally dual. \square

In the following section this result will be used in the construction of the reflection functors. For instance iterated one-point extensions allow us to pass from the base case of $q: [2]^n \rightarrow R^n$ studied in §7 to the more complicated case in the context of arbitrary oriented trees.

9. ABSTRACT TILTING THEORY FOR TREES

In this section we construct an abstract version of reflection functors for trees and show that they induce strongly stably equivalent quivers. As an application we obtain an abstract tilting result for trees (see Theorem 9.11, its discussion, and its corollaries). We carry out the details of the strategy outlined in §5. Thus, let Q be an oriented tree, let $q_0 \in Q$ be a source of valence n , and let $f_i: q_0 \rightarrow q_i, 1 \leq i \leq n$, be the morphisms adjacent to q_0 .

We begin with step (i) of the strategy mentioned in §5 and hence pass to the category Q_1 which is obtained by gluing in an n -cube $[2]^n$ of length two. To make this precise let us recall the cone construction. Given a small category A , then the **cone** A^\triangleleft is the small category obtained from A by adjoining a new initial object $-\infty$. The cone construction is obviously functorial in A and it comes with a fully faithful natural inclusion functor $i_A: A \rightarrow A^\triangleleft$. In particular, this gives rise to the commutative diagram of small categories

$$\begin{array}{ccccc} [1]_{=n-1}^n & \longrightarrow & [1]_{\geq n-1}^n & \longrightarrow & [1]^n \\ \downarrow & & \downarrow & & \downarrow \\ ([1]_{=n-1}^n)^\triangleleft & \longrightarrow & ([1]_{\geq n-1}^n)^\triangleleft & \longrightarrow & ([1]^n)^\triangleleft. \end{array}$$

(Note that these squares are not pushout squares in \mathcal{Cat} .) We observe that the morphisms $f_i: q_0 \rightarrow q_i, 1 \leq i \leq n$, together define a functor $([1]_{=n-1}^n)^\triangleleft \rightarrow Q$. This together with the construction (4.6) of finite biproduct diagrams via n -cubes gives

rise to the following commutative diagram of small categories

$$(9.1) \quad \begin{array}{ccccccccc} [1]_{=n-1}^n & \longrightarrow & [1]_{\geq n-1}^n & \longrightarrow & [1]^n & \longrightarrow & I & \longrightarrow & [2]^n \\ \downarrow & & \downarrow & & \downarrow & & \downarrow i^{(3)} & & \downarrow \iota_1 \\ ([1]_{=n-1}^n)^\triangleleft & \longrightarrow & ([1]_{\geq n-1}^n)^\triangleleft & \longrightarrow & ([1]^n)^\triangleleft & & & & \\ \downarrow & & \downarrow & & \downarrow & & & & \\ Q & \xrightarrow{u_1} & Q^{(1)} & \xrightarrow{u_2} & Q^{(2)} & \xrightarrow{u_3} & Q^{(3)} & \xrightarrow{u_4} & Q_1, \end{array}$$

in which the four additional squares are pushout squares. Note that the top row is precisely (4.6). It is easy to check that the functors $u_j, j = 1, \dots, 4$, in the bottom row are fully faithful, so that also the Kan extension functors

$$(9.2) \quad \mathcal{D}Q \xrightarrow{(u_1)_*} \mathcal{D}Q^{(1)} \xrightarrow{(u_2)_*} \mathcal{D}Q^{(2)} \xrightarrow{(u_3)!} \mathcal{D}Q^{(3)} \xrightarrow{(u_4)_*} \mathcal{D}Q_1$$

are fully faithful. For a stable derivator \mathcal{D} , recall from Proposition 4.9 the definition of the derivator $\mathcal{D}^{[2]^n, \text{ex}}$ of coherent biproduct diagrams based on (non-invertible) n -cubes, and note also that the functor $\iota_1: [2]^n \rightarrow Q_1$ is defined in (9.1).

Proposition 9.3. *Let \mathcal{D} be a stable derivator. Then (9.2) induces an equivalence between \mathcal{D}^Q and the full subderivator $\mathcal{D}^{Q_1, \text{ex}}$ of \mathcal{D}^{Q_1} spanned by all diagrams X such that $\iota_1^*(X)$ lies in $\mathcal{D}^{[2]^n, \text{ex}}$. This equivalence is natural with respect to exact morphisms of stable derivators.*

Proof. Since the functors in (9.2) are fully faithful, their composition induces an equivalence onto the essential image. Once we know that this essential image is precisely $\mathcal{D}^{Q_1, \text{ex}}$ this justifies that $\mathcal{D}^{Q_1, \text{ex}}$ actually is a derivator. This equivalence is natural with respect to exact morphisms of stable derivators because only homotopy finite Kan extensions are involved (see [GŠ14, Theorem 4.6]). We now go through the individual steps of the construction and justify that they do what they are supposed to, i.e., that the additional branches of the tree attached to the q_i do not perturb this construction.

The first step $(u_1)_*$ adds only one new object to our coherent diagrams which will become the final vertex of the n -cube $[2]^n$. Since $u_1: Q \rightarrow Q^{(1)}$ is the inclusion of a sieve, it follows that $(u_1)_*$ is right extension by zero, and hence induces a natural equivalence between \mathcal{D}^Q and the full subderivator $\mathcal{D}^{Q^{(1)}, \text{ex}}$ of $\mathcal{D}^{Q^{(1)}}$ spanned by all diagrams vanishing on the new object (see Lemma 3.2).

The second step $(u_2)_*$ is also fully faithful, and hence induces an equivalence onto its essential image. Since the functor u_2 is obtained from the fully faithful inclusion $v_2: ([1]_{\geq n-1}^n)^\triangleleft \rightarrow ([1]^n)^\triangleleft$ in (9.1) by finitely many one-point extensions, Theorem 8.8(ii) implies that X lies in the essential image of $(u_2)_*$ if and only if the restriction of X to $([1]^n)^\triangleleft$ lies in the essential image of $(v_2)_*$. In other words, writing $i^{(2)}: [1]^n \rightarrow Q^{(2)}$ for the functor in (9.1), it is easy to see that the essential image of $(u_2)_*$ is the full subderivator of $\mathcal{D}^{Q^{(2)}}$ spanned by the diagrams X such that $(i^{(2)})^*X$ is strongly bicartesian. Let us write $\mathcal{D}^{Q^{(2)}, \text{ex}}$ for the full subderivator of $\mathcal{D}^{Q^{(2)}}$ of the diagrams X such that $(i^{(2)})^*X$ is strongly bicartesian and vanishes on the final vertex. Then, this second step induces a natural equivalence $\mathcal{D}^{Q^{(1)}, \text{ex}} \simeq \mathcal{D}^{Q^{(2)}, \text{ex}}$.

The functor $u_3: Q^{(2)} \rightarrow Q^{(3)}$ is the inclusion of a cosieve and $(u_3)!$ is hence left extension by zero. Thus, by Lemma 3.2, $(u_3)!: \mathcal{D}^{Q^{(2)}} \rightarrow \mathcal{D}^{Q^{(3)}}$ induces an

equivalence onto the full subderivator of $\mathcal{D}^{Q^{(3)}}$ spanned by all objects X such that $(i^{(3)})^*(X) \in \mathcal{D}^I$ vanishes on the objects $(0, 2, \dots, 2), \dots, (2, \dots, 2, 0)$ (the functor $i^{(3)}$ is defined via (9.1)). If we write $\mathcal{D}^{Q^{(3)}, \text{ex}}$ for the full subderivator of $\mathcal{D}^{Q^{(3)}}$ spanned by the diagrams X such that $(i^{(3)})^*(X)$ satisfies these vanishing conditions, makes the n -cube bicartesian, and also vanishes on the final object, then $(u_3)!$ induces a natural equivalence $\mathcal{D}^{Q^{(2)}, \text{ex}} \simeq \mathcal{D}^{Q^{(3)}, \text{ex}}$.

It remains to study the final step $(u_4)_*: \mathcal{D}^{Q^{(3)}} \rightarrow \mathcal{D}^{Q_1}$. The claim is that this functor amounts to adding strongly bicartesian cubes everywhere, and that $(u_4)_*$ hence induces an equivalence onto the corresponding full subderivator of \mathcal{D}^{Q_1} . The details of the proof of this claim are basically a combination of the details of the proof of Proposition 4.9 together with the observation that the additional branches of the tree attached to the q_i can be ignored. To put this latter claim a bit more precisely, u_4 is obtained from the inclusion $w_4: I \rightarrow [2]^n$ in (9.1) by finitely many one-point extensions, where we add q_0 first and then all the branches at q_1, \dots, q_n of Q which do not contain q_0 . This way, we can again use Theorem 8.8(ii) to infer that $X \in \mathcal{D}^{Q_1}$ is in the essential image of $(u_4)_*$ if and only if the restriction of X to $[2]^n$ is in the essential image of $(w_4)_*$, and then conclude by the proof of Proposition 4.9.

As a summary, (9.2) induces natural equivalences of stable derivators

$$\mathcal{D}^Q \xrightarrow[(u_1)_*]{\simeq} \mathcal{D}^{Q^{(1)}, \text{ex}} \xrightarrow[(u_2)_*]{\simeq} \mathcal{D}^{Q^{(2)}, \text{ex}} \xrightarrow[(u_3)!]{\simeq} \mathcal{D}^{Q^{(3)}, \text{ex}} \xrightarrow[(u_4)_*]{\simeq} \mathcal{D}^{Q_1, \text{ex}},$$

and it is immediate that the final derivator $\mathcal{D}^{Q_1, \text{ex}}$ is precisely the one considered in the statement of this proposition. \square

We now take care of steps (ii) and (iii) of the strategy outlined in §5. This basically amounts to recycling the results of §7 and §8. More specifically, we pass from the category Q_1 to the category Q_2 obtained by ‘inverting the n -cube’, i.e., we thus consider the pushout

$$(9.4) \quad \begin{array}{ccc} [2]^n & \xrightarrow{q} & R^n \\ \downarrow \iota_1 & & \downarrow \iota_2 \\ Q_1 & \xrightarrow{u_5} & Q_2, \end{array}$$

where q is the localization functor studied in §7 (see Corollary 7.5). Observe that u_5 is obtained from q by iterated one-point extensions.

- (i) We first form a one-point extension of $[2]^n$ by identifying $1 \in [1]$ with the center $(1, 1, \dots, 1)$ of the n -cube $[2]^n$, which amounts to adding the source q_0 of the quiver.
- (ii) As the morphisms $q_0 \rightarrow q_i$ are already taken care of, we next have to add all the additional branches of the tree which are attached to the q_i . This can be done inductively over the numbers $1 \leq i \leq n$ and then inductively over the number of branches and then over the lengths of these branches by means of one-point extensions only. That way we obtain the functor $\iota_1: [2]^n \rightarrow Q_1$.
- (iii) Applying the same one-point extensions to the functor $q: [2]^n \rightarrow R^n$ we obtain the localization functor $u_5: Q_1 \rightarrow Q_2$ in (9.4).

Proposition 9.5. *The functor $u_5: Q_1 \rightarrow Q_2$ is a homotopical epimorphism. For a derivator \mathcal{D} , $u_5^*: \mathcal{D}^{Q_2} \rightarrow \mathcal{D}^{Q_1}$ induces an equivalence onto the full subderivator of \mathcal{D}^{Q_1} spanned by all diagrams X such that in $\iota_1^*(X) \in \mathcal{D}^{[2]^n}$ the morphisms*

$$(i_1, \dots, i_{k-1}, 0, i_{k+1}, \dots, i_n) \rightarrow (i_1, \dots, i_{k-1}, 2, i_{k+1}, \dots, i_n)$$

are sent to isomorphisms for all $i_1, \dots, i_{k-1}, i_{k+1}, \dots, i_n$ and k .

Proof. We know that $q: [2]^n \rightarrow R^n$ is a homotopical epimorphism and that its essential image admits a similar description (see [Corollary 7.5](#)). The above description of u_5 as an iterated one-point extension of q together with [Proposition 8.6](#) implies that u_5 is also a homotopical epimorphism. Using the description of the essential image of q^* given in [Corollary 7.5](#), an inductive application of [Theorem 8.8\(i\)](#) gives us the desired description of the essential image of $(u_5)^*$. \square

For a stable derivator \mathcal{D} , recall the definition of $\mathcal{D}^{Q_1, \text{ex}}$ in [Proposition 9.3](#) and of the derivator $\mathcal{D}^{R^n, \text{ex}}$ of coherent biproduct diagrams based on invertible n -cubes as given just prior to [Corollary 7.6](#). Let us denote by $\mathcal{D}^{Q_2, \text{ex}}$ the full subderivator of \mathcal{D}^{Q_2} spanned by the diagrams X such that $\iota_2^*(X)$ lies in $\mathcal{D}^{R^n, \text{ex}}$ (the functor $\iota_2: R^n \rightarrow Q_2$ is defined via (9.4)).

Corollary 9.6. *Let \mathcal{D} be a stable derivator and let $u_5: Q_1 \rightarrow Q_2$ be as in (9.4). Then $u_5^*: \mathcal{D}^{Q_2} \rightarrow \mathcal{D}^{Q_1}$ induces a natural equivalence $\mathcal{D}^{Q_2, \text{ex}} \rightarrow \mathcal{D}^{Q_1, \text{ex}}$.*

Proof. We saw at the end of §7 that $q^*: \mathcal{D}^{R^n} \rightarrow \mathcal{D}^{[1]^n}$ induces a natural equivalence of stable derivators $\mathcal{D}^{R^n, \text{ex}} \rightarrow \mathcal{D}^{[1]^n, \text{ex}}$. It is immediate that the equivalence of [Proposition 9.5](#) restricts to the desired natural equivalence $\mathcal{D}^{Q_2, \text{ex}} \rightarrow \mathcal{D}^{Q_1, \text{ex}}$. \square

We turn to the final step (iv) of the strategy described in §5. In this step we add the cofiber of the map $q_0 \rightarrow (1, \dots, 1)$ in Q_2 , where $(1, \dots, 1)$ is the object supporting the biproduct. Recall from (3.3) our naming convention for the objects in $\square = [1]^2$. Let $[1] \rightarrow \ulcorner$ classify the horizontal map $(0, 0) \rightarrow (1, 0)$ and let $\ulcorner \rightarrow \square$ be the obvious inclusion. We consider the pushout squares

$$(9.7) \quad \begin{array}{ccccc} [1] & \xrightarrow{\quad} & \ulcorner & \xrightarrow{\quad} & \square \\ \downarrow & & \downarrow i^{(4)} & & \downarrow \iota_3 \\ Q_2 & \xrightarrow{u_6} & Q^{(4)} & \xrightarrow{u_7} & Q_3, \end{array}$$

in which the vertical map on the left classifies the arrow $q_0 \rightarrow (1, \dots, 1)$ in Q_2 . We denote the objects in the image of ι_3 by

$$\begin{array}{ccc} q_0 & \xrightarrow{\quad} & b \\ \downarrow & & \downarrow \\ z & \xrightarrow{\quad} & c, \end{array}$$

where the names in turn stand for biproduct, zero, and cofiber. If \mathcal{D} is a derivator, then the Kan extension functors

$$(9.8) \quad \mathcal{D}^{Q_2} \xrightarrow{(u_6)_*} \mathcal{D}^{Q^{(4)}} \xrightarrow{(u_7)!} \mathcal{D}^{Q_3}$$

are fully faithful as the same is true for u_6 and u_7 .

Proposition 9.9. *If \mathcal{D} is pointed, then (9.8) induces an equivalence between \mathcal{D}^{Q_2} and the full subderivator of \mathcal{D}^{Q_3} spanned by all X such that $\iota_3^*(X) \in \mathcal{D}^\square$ is a cofiber square, i.e., $\iota_3^*(X) \in \mathcal{D}^\square$ is cocartesian and vanishes on $(0, 1) \in \square$.*

Proof. It is enough to describe the essential images of $(u_6)_*$ and $(u_7)_!$. As u_6 is the inclusion of a sieve, it follows that $(u_6)_*$ is right extension by zero. Thus, Lemma 3.2 implies that $(u_6)_*: \mathcal{D}^{Q_2} \rightarrow \mathcal{D}^{Q^{(4)}}$ induces an equivalence onto the full subderivator of $\mathcal{D}^{Q^{(4)}}$ spanned by all X such that $i^{(4)*}(X)$ vanishes at $(0, 1)$ (where $i^{(4)}$ is defined via (9.7)).

We now turn to the functor $(u_7)_!: \mathcal{D}^{Q^{(4)}} \rightarrow \mathcal{D}^{Q_3}$. Since u_7 is fully faithful and the complement of the image consists of precisely the object c only, we can detect the essential image of $(u_7)_!$ by considering the counit at c ([Gro13, Lemma 1.21]). Thus, X lies in the essential image of $(u_7)_!$ if and only if $\epsilon_c: (u_7)_! u_7^*(X)_c \rightarrow X_c$ is an isomorphism. To analyze this further we consider the pasting

$$\begin{array}{ccccccc} \ulcorner & \xrightarrow{r} & (u_7/c) & \longrightarrow & Q^{(4)} & \xrightarrow{u_7} & Q_3 \\ \downarrow & & \downarrow & \nearrow & \downarrow u_7 & & \downarrow = \\ \mathbb{1} & \longrightarrow & \mathbb{1} & \xrightarrow{c} & Q_3 & \xrightarrow{=} & Q_3, \end{array}$$

where $r: \ulcorner \rightarrow (u_7/c)$ is given by

$$\begin{array}{ccc} q_0 & \longrightarrow & b \\ \downarrow & \searrow & \downarrow \\ z & \longrightarrow & c. \end{array}$$

Let us assume for the moment that r is a right adjoint. Using the compatibility of mates with pasting, (Der4), and the homotopy finality of right adjoints (see Examples 2.7) we then conclude that ϵ_c is an isomorphism if and only if the mate of the above pasting is an isomorphism. But this pasting can be rewritten as

$$\begin{array}{ccccccc} \ulcorner & \xrightarrow{\cong} & (i_\ulcorner/(1, 1)) & \longrightarrow & \ulcorner & \xrightarrow{i_\ulcorner} & [1]^2 \xrightarrow{\iota_3} Q_3 \\ \downarrow & & \downarrow & \nearrow & \downarrow i_\ulcorner & & \downarrow = \\ \mathbb{1} & \xrightarrow{=} & \mathbb{1} & \xrightarrow{(1, 1)} & [1]^2 & \xrightarrow{=} & [1]^2 \xrightarrow{\iota_3} Q_3, \end{array}$$

where ι_3 is defined via (9.7). Thus, using similar arguments, we deduce that ϵ_c is an isomorphism if and only if the counit $(i_\ulcorner)_!(i_\ulcorner)^* \iota_3^*(X) \rightarrow \iota_3^*(X)$ is an isomorphism when evaluated at $(1, 1)$. By a further application of [Gro13, Lemma 1.21] we see that this is the case if and only if $\iota_3^*(X)$ is cocartesian.

Hence, assuming that r indeed is a right adjoint, we see that $(u_7)_!: \mathcal{D}^{Q^{(4)}} \rightarrow \mathcal{D}^{Q_3}$ induces an equivalence onto the full subderivator of \mathcal{D}^{Q_3} of those X such that $\iota_3^*(X)$ is cocartesian. It is easy to see that this restricts to a further equivalence between the respective full subderivators of objects vanishing at z . Putting these two steps together gives the desired result.

As for the existence of the left adjoint to $r: \ulcorner \rightarrow (u_7/c)$, let us consider objects of (u_7/c) which do not lie in the image of r . These are pairs $(x, u_7(x) \rightarrow c)$ such that the structure map factors uniquely through $b \rightarrow c$. We leave it to the reader

to check that the assignment which sends all such objects to $(1, 0)$ and which is inverse to r on the remaining objects defines a left adjoint $l: (u_7/c) \rightarrow \ulcorner$ to r . \square

In order to formulate the following corollary and as a preparation for the proof of [Theorem 9.11](#), let us summarize in the following diagram part of the categories involved in the constructions so far (as spelled out in [\(9.1\)](#), [\(9.4\)](#), and [\(9.7\)](#)),

$$\begin{array}{ccccc} & [2]^n & \longrightarrow & R^n & \square \\ & \downarrow \iota_1 & & \downarrow \iota_2 & \downarrow \iota_3 \\ Q & \longrightarrow & Q_1 & \longrightarrow & Q_2 \longrightarrow Q_3. \end{array}$$

If \mathcal{D} is a stable derivator, then let $\mathcal{D}^{Q_3, \text{ex}}$ be the full subderivator of \mathcal{D}^{Q_3} spanned by all diagrams X such that the restriction to Q_2 lies in $\mathcal{D}^{Q_2, \text{ex}}$ and $\iota_3^*(X) \in \mathcal{D}^\square$ is a cofiber square.

Corollary 9.10. *Let \mathcal{D} be a stable derivator. Then [\(9.8\)](#) induces a natural equivalence $\mathcal{D}^{Q_2, \text{ex}} \rightarrow \mathcal{D}^{Q_3, \text{ex}}$.*

Proof. This follows immediately from [Proposition 9.9](#) because the functors in [\(9.8\)](#) are fully faithful. \square

We now only have to assemble the results obtained so far in order to prove our main result.

Theorem 9.11. *Let Q be an oriented tree, let $q_0 \in Q$ be a source, and let $Q' = \sigma_{q_0} Q$ be the reflected quiver. The quivers Q and Q' are strongly stably equivalent.*

Proof. Let \mathcal{D} be a stable derivator. Then by [Proposition 9.3](#), [Corollary 9.6](#), and [Corollary 9.10](#) there are natural equivalences of stable derivators

$$\mathcal{D}^Q \xrightarrow{\sim} \mathcal{D}^{Q_1, \text{ex}} \xleftarrow{\sim} \mathcal{D}^{Q_2, \text{ex}} \xrightarrow{\sim} \mathcal{D}^{Q_3, \text{ex}}.$$

If we instead begin with the reflected quiver Q' and perform the dual constructions, then we obtain a similarly defined –actually dual– sequence of natural equivalences of stable derivators

$$\mathcal{D}^{Q'} \xrightarrow{\sim} \mathcal{D}^{Q'_1, \text{ex}} \xleftarrow{\sim} \mathcal{D}^{Q'_2, \text{ex}} \xrightarrow{\sim} \mathcal{D}^{Q'_3, \text{ex}}.$$

Note that the category Q_3 contains the quiver Q and the reflected quiver Q' as subcategories and that this category is completely symmetric. Said differently, things can be set up in a way that $Q_3 = Q'_3$ and, using the stability of \mathcal{D} , that also the derivators $\mathcal{D}^{Q_3, \text{ex}}$ and $\mathcal{D}^{Q'_3, \text{ex}}$ agree. Thus, we obtain a sequence of natural equivalences

$$\mathcal{D}^Q \xrightarrow{\sim} \mathcal{D}^{Q_1, \text{ex}} \xleftarrow{\sim} \mathcal{D}^{Q_2, \text{ex}} \xrightarrow{\sim} \mathcal{D}^{Q_3, \text{ex}} \stackrel{!}{=} \mathcal{D}^{Q'_3, \text{ex}} \xleftarrow{\sim} \mathcal{D}^{Q'_2, \text{ex}} \xrightarrow{\sim} \mathcal{D}^{Q'_1, \text{ex}} \xleftarrow{\sim} \mathcal{D}^{Q'},$$

which is to say that Q and Q' are strongly stably equivalent. \square

Definition 9.12. Let \mathcal{D} be a stable derivator. The components $s_{q_0}^-: \mathcal{D}^Q \xrightarrow{\sim} \mathcal{D}^{Q'}$ of the strong stable equivalence $s_{q_0}^-: Q \stackrel{s}{\sim} Q'$ constructed in the proof of [Theorem 9.11](#) are the **reflection functors** associated to the oriented tree Q and the source q_0 . The inverse strong stable equivalence $s_{q_0}^+: Q' \stackrel{s}{\sim} Q$ has components $s_{q_0}^+: \mathcal{D}^{Q'} \xrightarrow{\sim} \mathcal{D}^Q$, which are also called reflection functors.

Example 9.13. Let k be a field and let \mathcal{D}_k be the derivator of k , i.e., the homotopy derivator associated to the projective model structure on unbounded chain complexes over k . Recall that there are equivalences $\mathcal{D}_k^Q \simeq \mathcal{D}_{kQ}$ and the reflection functors $(s_{q_0}^-, s_{q_0}^+): Q \xrightarrow{s} Q'$ hence specialize to an equivalence of derivators

$$(s_{q_0}^-, s_{q_0}^+): \mathcal{D}_{kQ} \rightleftarrows \mathcal{D}_{kQ'}.$$

The underlying exact equivalence of triangulated categories $D(kQ) \simeq D(kQ')$ can be identified with the equivalence in [Theorem 5.3](#) established by Happel in [\[Hap86\]](#).

Since these reflection functors are available for *every stable derivator*, we immediately get similar equivalences for the derivator \mathcal{D}_R of a ring R , for the derivator \mathcal{D}_X of a (quasi-compact and quasi-separated) scheme X , for the derivator \mathcal{D}_A of a differential-graded algebra A , for the derivator \mathcal{D}_E of a (symmetric) ring spectrum E , and for further examples arising in algebra, geometry, and topology. Moreover, these reflection functors are natural with respect to exact morphisms of derivators. For more details on further examples of stable derivators we refer to [\[GŠ14, §5\]](#). Also the following corollaries have implications for all these different contexts.

Corollary 9.14. *Two oriented trees are strongly stably equivalent if and only if the underlying unoriented graphs are isomorphic.*

Proof. It is a purely combinatorial argument that arbitrary reorientations of trees can be obtained by iterated reflections at sources and sinks; see [\[BGP73, Theorem 1.2\(1\)\]](#). Thus, different orientations of the same tree are strongly stably equivalent by an iterated use of [Theorem 9.11](#). If on the other hand Q, Q' are two oriented trees and \mathcal{D}_k is the derivator of any field k , then \mathcal{D}_k^Q and $\mathcal{D}_k^{Q'}$ can only be equivalent if the underlying graphs of Q and Q' are isomorphic by [\[GŠ14, Proposition 5.4\]](#). \square

Let us agree that a **forest** is a disjoint union of trees.

Corollary 9.15. *Let F_1 and F_2 be not necessarily connected quivers with the same forest as underlying graph. Then F_1 and F_2 are strongly stably equivalent.*

Proof. This is immediate from [Corollary 9.14](#) and (Der1). \square

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